Pythagorean theorem proofs and applications philosophy essay

Introduction

There are an uncountable number of topics that students are expected to cover each year in school. For example, they are expected to learn about right triangles, similar triangles, and polygons. We expect them to learn about angles, lines, and graphs. One of the topics that almost every high school geometry student learns about is the Pythagorean Theorem.

When asked what the Pythagorean Theorem is, students will often state that a2+b2= c2 where a, b, and c are sides of a right triangle. However, students often don't know why this is true. Most have never proved it. On the pages that follow, there is history, several proofs, and ways for high school students to use the proof in real life situations.

To understand the following information, a general knowledge of geometry and algebra should be sufficient. The proofs are not difficult and with some thought it is clear that the Pythagorean Theorem works and is important.

The Pythagorean Theorem

According to the Pythagorean Theorem, " In a right triangle, the square of the hypotenuse is equal to the sum of the square of the other two sides of the triangle." The theorem can be written as an equation relating the lengths of the sides a, b and c, often called the Pythagorean equation

In terms of area, it states:

In any right triangle, the area of the square whose side is

the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle).

In a simple example of how the Pythagorean Theorem might be used, someone might be wondering about how long it would take to cut across a rectangular plot of land, rather than skirting the edges, relying on the principle that a rectangle can be divided into two simple right triangles. He or she could measure two adjoining sides, determine their squares, add the squares together, and find the square root of the sum to determine the length of the plot's diagonal.

Like other mathematical theorems, the Pythagorean Theorem relies on proofs. Each proof is designed to create more supporting evidence to show that the theorem is correct, by demonstrating various applications, showing the shapes that the Pythagorean Theorem cannot be applied to, and attempting to disprove the Pythagorean Theorem to show, in reverse, that the logic behind the theorem is found. Because the Pythagorean Theorem is one of the oldest math theorems in use today, it is also one of the most heavily proved, with hundreds of proofs by mathematicians throughout history adding to the body of evidence which shows that the theorem is valid.

The Pythagorean Theorem is one of the earliest known theorems to ancient civilizations. It was named after Pythagoras, a Greek mathematician and philosopher. The theorem bears his name although we have evidence that the Babylonians knew this relationship some 1000 years earlier. (Plimpton 322), a Babylonian mathematical tablet dated back to 1900 B. C., contains a table of Pythagorean triples. Later in the essay it is explained what are Pythagorean Triples and how they work. The Chou-pei, an ancient Chinese text, also gives us evidence that the Chinese knew about the Pythagorean Theorem many years before Pythagoras or one of his colleagues in the Pythagorean society discovered and proved it. This is the reason why the theorem is named after Pythagoras. It is possible that someone proved the theorem before Pythagoras, but no proof has been found.

After Babylonians' method of using this theorem, about thousand years later, between the years of 580-500 BC, Pythagoras of Samos was the first to prove the theorem. Because of this, Pythagoras is given credit for the first proof. (MacTutor History of Mathematics Archive, 2010)

Proofs of Pythagorean Theorem

As mentioned earlier, the theorem states that:

In terms of areas, it states:

In any right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs. This theorem is talking about the area of the squares that are built on each side of the right triangle.

Accordingly, we obtain the following areas for the squares, where the green and blue squares are on the legs of the right triangle and the red square is on the hypotenuse.

Area of the green square is

Area of the blue square is

Area of the red square is

From our theorem, we have the following relationship:

Area of green square + Area of blue square = Area of red square or

Pythagoras' Proof

" Let a, b, c denote the legs and the hypotenuse of the given right triangle, and consider the two squares in the accompanying fiqure, each having a+b as its side. The first square is dissected into six pieces-namely, the two squares on the legs and four right triangles congruent to the given triangle. The second square is dissected into five pieces-namely, the square on the hypotenuse and four right triangles congruent to the given triangle. By subtracting equals from equals, it now follows that the square on the hypotenuse is equal to the sum of the squares on the legs" (Eves 81).

Consider the following figure.

The area of the first square is given by (a+b)^2 or 4(1/2ab)+ a^2 + b^2.

The area of the second square is given by (a+b)^2 or 4(1/2ab) + c^2.

Since the squares have equal areas we can set them equal to another and subtract equals. The case (a+b)^2=(a+b)^2 is not interesting. Let's do the other case.

4(1/2ab) + a^2 + b^2 = 4(1/2ab)+ c^2

Subtracting equals from both sides we have

This concludes Pythagoras' proof of the theorem.

Over the years there have been many mathematicians and non-mathematicians to give various proofs of the Pythagorean Theorem. Following are proofs from Bhaskara and one of our former presidents, President James Garfield. I have chosen these proofs because any of them would be appropriate to use in any classroom.

Bhaskara's First Proof of the Pythagorean Theorem

Bhaskara's proof is also a dissection proof. It is similar to the proof provided by Pythagoras. Bhaskara was born in India. He was one of the most important Hindu mathematicians of the second century AD. He used the following diagrams in proving the Pythagorean Theorem.

In the above diagrams, the blue triangles are all congruent and the yellow squares are congruent. First we need to find the area of the big square two different ways. First let's find the area using the area formula for a square.

Thus, A= c^2.

Now, lets find the area by finding the area of each of the components and then sum the areas.

Area of the blue triangles = 4(1/2)ab

Area of the yellow square = (b-a)^2

Area of the big square = 4(1/2)ab + (b-a)^2

$= 2ab + b^2 – 2ab + a^2$

$= b^2 + a^2$

Since, the square has the same area no matter how you find it

$A = c^2 = a^2 + b^2$,

concluding the proof.

AA Postulate

Before solving another proof for Pythagorean Theorem, I want to state ' AA Postulate' from Euclid's Geometry.

Theorem:

If an angle of one triangle is congruent to an angle of another triangle and the lengths of the sides including those angles are proportional, the triangles are similar.

Bhaskara's Second Proof of the Pythagorean Theorem

In this proof, Bhaskara began with a right triangle and then he drew an altitude on the hypotenuse. From here, he used the properties of similarity to prove the theorem.

Now prove that triangles ABC and CBE are congruent.

It follows from the AA postulate that triangle ABC is similar to triangle CBE, since angle B is congruent to angle B and angle C is congruent to angle E. Thus, since internal ratios are equal s/a= a/c

Multiplying both sides by ac we get

$sc = a^2$

Now show that triangles ABC and ACE are similar.

As before, it follows from the AA postulate that these two triangles are similar. Angle A is congruent to angle A and angle C is congruent to angle E. Thus, $r/b = b/c$. Multiplying both sides by bc we get

$rc = b^2$.

Now when we add the two results we get

$sc + rc = a^2 + b^2$.

$c(s+r) = a^2 + b^2$

$c^2 = a^2 + b^2$,

concluding the proof of the Pythagorean Theorem.

Garfield’s Proof

The twentieth president of the United States gave the following proof to the Pythagorean Theorem. He discovered this proof five years before he become President. He hit upon this proof in 1876 during a mathematics discussion with some of the members of Congress. It was later published in the New England Journal of Education. The proof depends on calculating the area of a right trapezoid two different ways. The first way is by using the area formula of a trapezoid and the second is by suming up the areas of the three right triangles that can be constructed in the trapezoid. He used the following trapezoid in developing his proof.

First, we need to find the area of the trapezoid by using the area formula of the trapezoid.

$A=(1/2)h(b1+b2)$ area of a trapezoid

In the above diagram, $h= a+b$, $b1= a$, and $b2= b$.

$A=(1/2)(a+b)(a+b)$

$=(1/2)(a^2+2ab+b^2)$.

Now, let's find the area of the trapezoid by summing the area of the three right triangles.

The area of the yellow triangle is

$A= 1/2(ba)$.

The area of the red triangle is

$A= 1/2(c^2)$.

The area of the blue triangle is

$A= 1/2(ab)$.

The sum of the area of the triangles is

$1/2(ba) + 1/2(c^2) + 1/2(ab) = 1/2(ba + c^2 + ab) = 1/2(2ab + c^2)$.

Since, this area is equal to the area of the trapezoid we have the following relation:

$(1/2)(a^2 + 2ab + b^2) = (1/2)(2ab + c^2)$.

Multiplying both sides by 2 and subtracting 2ab from both sides we get

This concludes another proof for Pythagorean Theorem.

Pythagorean triplet

Before a proof was ever given for the Pythagorean theorem, besides the Babylonians it was thought that “ Egyptian temple builders used ropes in laying foundations, suggested that perhaps the obtained accurate right angles by using marked ropes that could be stretched around stakes to form a 3, 4, 5 right triangle.” There is no documented evidence of this but Cantor, a historian of mathematics agreed this could be true. (Gardner, 155) They had a list that contained all the Pythagorean Triples.

Some special shapes can be described with the Pythagorean Theorem. A Pythagorean triple is a right triangle in which the lengths of the sides and hypotenuse are all whole numbers. The smallest Pythagorean triple is a triangle in which a= 3, b= 4, and c= 5. Using the Pythagorean theorem, people can see that 9+16= 25. The squares in the theorem can also be literal; if one were to use every length of a right triangle as the side of a square, the squares of the sides would have the same area as the square created by the length of the hypotenuse.

A Pythagorean triple consists of three positive integers a, b, and c, such that a2 + b2 = c2. Such a triple is commonly written (a, b, c), and a well-known example is (3, 4, 5). In other words, a Pythagorean triple represents the lengths of the sides of a right triangle where all three sides have integer lengths

Euclid's formula[1] is a fundamental formula for generating Pythagorean triples given an arbitrary pair of positive integers m and n with m > n. The formula states that the integers

form a Pythagorean triple.

Applications of Pythagorean Theorem –

Use Any Shape –

We used triangles in our diagram, the simplest 2-D shape. But the line segment can belong to any shape. I found this by using circles, for example:

Now what happens when I add them together?

Circle of radius 5 = Circle of radius 4 + Circle of radius 3

Hence there area of the big circle with radius 5 ƒ

25Ï€ = 16Ï€ + 9Ï€

â‰ˆ 78. 54 = â‰ˆ 50. 27 + â‰ˆ 28. 27

â‰ˆ 78. 54 = â‰ˆ 78. 54

We can multiply the Pythagorean Theorem by our area factor (pi (Ï€), in this case) and come up with a relationship for any shape. The line segment can be any portion of the shape. I could have picked the circle's radius, diameter, or circumference – there would be a different area factor, but the 3-4-5 relationship would still hold.

So, whether you're adding up pizzas or Richard Nixon masks, the Pythagorean Theorem helps you relate the areas of any similar shapes.

Pythagorean Theorem's uses in 3-D

In terms of solid geometry, Pythagoras' theorem can be applied to three dimensions as follows. Consider a rectangular solid as shown in the figure. The length of diagonal BD is found from Pythagoras' theorem as:

where, these three sides form a right triangle. Using horizontal diagonal BD and the vertical edge AB, the length of diagonal AD then is found by a second application of Pythagoras' theorem as:

or, doing it all in one step:

This one-step formulation may be viewed as a generalization of Pythagoras' theorem to higher dimensions. However, this result is really just the repeated application of the original Pythagoras' theorem to a succession of right triangles in a sequence of orthogonal planes.