Chapter 4

CHAPTER 4 PART II: VALUATION AND CAPITAL BUDGETING Discounted Cash Flow Valuation The signing of big-name athletes is often accompanied by great fanfare, but the numbers are often misleading. For example, in late 2010, catcher Victor Martinez reached a deal with the Detroit Tigers, signing a contract with a reported value of $50 million. Not bad, especially for someone who makes a living using the “ tools of ignorance" (jock jargon for a catcher’s equipment). Another example is the contract signed by Jayson Werth of the Washington Nationals, which had a stated value of $126 million. It looks like Victor and Jayson did pretty well, but then there was Carl Crawford, who signed to play in front of Boston’s Red Sox nation. Carl’s contract has a stated value of $142 million, but this amount was actually payable over several years. The contract consisted of a $6 million signing bonus, along with $14 million in the first year plus $122 million in future salary to be paid in the years 2011 through 2017. Victor’s and Jayson’s payments were similarly spread over time. Because all three contracts called for payments that are made at future dates, we must consider the time value of money, which means none of these players received the quoted amounts. How much did they really get? This chapter gives you the “ tools of knowledge" to answer this question. For updates on the latest happenings in finance, visit www . rwjcorporatefinance . blogspot. com 4. 1 Valuation: The One-Period Case Keith Vaughn is trying to sell a piece of raw land in Alaska. Yesterday he was offered $10, 000 for the property. He was about ready to accept the offer when another individual offered him $11, 424. However, the second offer was to be paid a year from now. Keith has satisfied himself that both buyers are honest and financially solvent, so he has no fear that the offer he selects will fall through. These two offers are pictured as cash flows in Figure 4. 1. Which offer should Keith choose? Mike Tuttle, Keith’s financial adviser, points out that if Keith takes the first offer, he could invest the $10, 000 in the bank at an insured rate of 12 percent. At the end of one year, he would have: $10, 000 1 (. 12 3 $10, 000) 5 $10, 000 3 1. 12 5 $11, 200 Return of Interest principal Because this is less than the $11, 424 Keith could receive from the second offer, Mike recommends that he take the latter. This analysis uses the concept of future value (FV) or compound value, which is the value of a sum after investing over one or more periods. The compound or future value of $10, 000 at 12 percent is $11, 200. 87 88 Part II Valuation and Capital Budgeting Figure 4. 1 Cash Flow for Keith Vaughn’s Sale Alternative sale prices $10, 000 $11, 424 Year: 0 1 An alternative method employs the concept of present value (PV). One can determine present value by asking the following question: How much money must Keith put in the bank today so that he will have $11, 424 next year? We can write this algebraically as: PV 3 1. 12 5 $11, 424 We want to solve for PV, the amount of money that yields $11, 424 if invested at 12 percent today. Solving for PV, we have: $11, 424 PV 5 ________ 5 $10, 200 1. 12 The formula for PV can be written as follows: Present Value of Investment: C1 PV 5 ______ 11r (4. 1) where C1 is cash flow at date 1 and r is the rate of return that Keith Vaughn requires on his land sale. It is sometimes referred to as the discount rate. Present value analysis tells us that a payment of $11, 424 to be received next year has a present value of $10, 200 today. In other words, at a 12 percent interest rate, Keith is indifferent between $10, 200 today or $11, 424 next year. If you gave him $10, 200 today, he could put it in the bank and receive $11, 424 next year. Because the second offer has a present value of $10, 200, whereas the first offer is for only $10, 000, present value analysis also indicates that Keith should take the second offer. In other words, both future value analysis and present value analysis lead to the same decision. As it turns out, present value analysis and future value analysis must always lead to the same decision. As simple as this example is, it contains the basic principles that we will be working with over the next few chapters. We now use another example to develop the concept of net present value. EXAMPLE 4. 1 Present Value Diane Badame, a financial analyst at Kaufman & Broad, a leading real estate firm, is thinking about recommending that Kaufman & Broad invest in a piece of land that costs $85, 000. She is certain that next year the land will be worth $91, 000, a sure $6, 000 gain. Given that the interest rate in similar alternative investments is 10 percent, should Kaufman & Broad undertake the investment in land? Diane's choice is described in Figure 4. 2 with the cash flow time chart. A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $85, 000 in the land, she will have $91, 000 available next year. Suppose, instead, Chapter 4 Discounted Cash Flow Valuation 89 Figure 4. 2 Cash Flows for Land Investment Cash inï¬, ow $91, 000 Time Cash outï¬, ow 0 1 2$85, 000 that Kaufman & Broad puts the same $85, 000 into similar alternative investments. At the interest rate of 10 percent, this $85, 000 would grow to: (1 1 . 10) 3 $85, 000 5 $93, 500 next year. It would be foolish to buy the land when investing the same $85, 000 in similar alternative investments would produce an extra $2, 500 (that is, $93, 500 from the bank minus $91, 000 from the land investment). This is a future value calculation. Alternatively, she could calculate the present value of the sale price next year as: $91, 000 Present value 5 ________ 5 $82, 727. 27 1. 10 Because the present value of next year's sales price is less than this year's purchase price of $85, 000, present value analysis also indicates that she should not recommend purchasing the property. Frequently, financial analysts want to determine the exact cost or benefit of a decision. In Example 4. 1, the decision to buy this year and sell next year can be evaluated as: âˆ'$2, 273 5 âˆ'$85, 000 Cost of land today 1 $91, 000 _______ 1. 10 Present value of next year's sales price The formula for NPV can be written as follows: Net Present Value of Investment: NPV 5 âˆ'Cost 1 PV (4. 2) Equation 4. 2 says that the value of the investment is âˆ'$2, 273, after stating all the benefits and all the costs as of date 0. We say that âˆ'$2, 273 is the net present value (NPV) of the investment. That is, NPV is the present value of future cash flows minus the present value of the cost of the investment. Because the net present value is negative, Lida Jennings should not recommend purchasing the land. Both the Vaughn and the Badame examples deal with a high degree of certainty. That is, Keith Vaughn knows with a high degree of certainty that he could sell his land for $11, 424 next year. Similarly, Diane Badame knows with a high degree of certainty that Kaufman & Broad could receive $91, 000 for selling its land. Unfortunately, businesspeople frequently do not know future cash flows. This uncertainty is treated in the next example. 90 Part II Valuation and Capital Budgeting EXAMPLE 4. 2 Uncertainty and Valuation Professional Artworks, Inc., is a firm that speculates in modern paintings. The manager is thinking of buying an original Picasso for $400, 000 with the intention of selling it at the end of one year. The manager expects that the painting will be worth $480, 000 in one year. The relevant cash flows are depicted in Figure 4. 3. Figure 4. 3 Cash Flows for Investment in Painting Expected cash inï¬, ow $480, 000 Time Cash outï¬, ow 0 1 2$400, 000 Of course, this is only an expectation–the painting could be worth more or less than $480, 000. Suppose the guaranteed interest rate granted by banks is 10 percent. Should the firm purchase the piece of art? Our first thought might be to discount at the interest rate, yielding: $480, 000 ________ 1. 10 5 $436, 364 Because $436, 364 is greater than $400, 000, it looks at first glance as if the painting should be purchased. However, 10 percent is the return one can earn on a low risk investment. Because the painting is quite risky, a higher discount rate is called for. The manager chooses a rate of 25 percent to reflect this risk. In other words, he argues that a 25 percent expected return is fair compensation for an investment as risky as this painting. The present value of the painting becomes: $480, 000 ________ 1. 25 5 $384, 000 Thus, the manager believes that the painting is currently overpriced at $400, 000 and does not make the purchase. The preceding analysis is typical of decision making in today's corporations, though real-world examples are, of course, much more complex. Unfortunately, any example with risk poses a problem not faced in a riskless example. Conceptually, the correct discount rate for an expected cash flow is the expected return available in the market on other investments of the same risk. This is the appropriate discount rate to apply because it represents an economic opportunity cost to investors. It is the expected return they will require before committing funding to a project. However, the selection of the discount rate for a risky investment is quite a difficult task. We simply don't know at this point whether the discount rate on the painting in Example 4. 2 should be 11 percent, 15 percent, 25 percent, or some other percentage. Because the choice of a discount rate is so difficult, we merely wanted to broach the subject here. We must wait until the specific material on risk and return is covered in later chapters before a risk-adjusted analysis can be presented. Chapter 4 Discounted Cash Flow Valuation 91 4. 2 The Multiperiod Case The previous section presented the calculation of future value and present value for one period only. We will now perform the calculations for the multiperiod case. FUTURE VALUE AND COMPOUNDING Suppose an individual were to make a loan of $1. At the end of the first year, the borrower would owe the lender the principal amount of $1 plus the interest on the loan at the interest rate of r. For the specific case where the interest rate is, say, 9 percent, the borrower owes the lender: $1 3 (1 1 r) 5 $1 3 1. 09 5 $1. 09 At the end of the year, though, the lender has two choices. She can either take the $1. 09–or, more generally, (1 1 r)–out of the financial market, or she can leave it in and lend it again for a second year. The process of leaving the money in the financial market and lending it for another year is called compounding. Suppose the lender decides to compound her loan for another year. She does this by taking the proceeds from her first one-year loan, $1. 09, and lending this amount for the next year. At the end of next year, then, the borrower will owe her: $1 3 (1 1 r) 3 (1 1 r) 5 $1 3 (1 1 r)2 5 1 1 2r 1 r2 $1 3 (1. 09) 3 (1. 09) 5 $1 3 (1. 09)2 5 $1 1 $. 18 1 $. 0081 5 $1. 1881 This is the total she will receive two years from now by compounding the loan. In other words, the capital market enables the investor, by providing a ready opportunity for lending, to transform $1 today into $1. 1881 at the end of two years. At the end of three years, the cash will be $1 3 (1. 09)3 5 $1. 2950. The most important point to notice is that the total amount the lender receives is not just the $1 that she lent plus two years' worth of interest on $1: 2 3 r 5 2 3 $. 09 5 $. 18 The lender also gets back an amount r2, which is the interest in the second year on the interest that was earned in the first year. The term 2 3 r represents simple interest over the two years, and the term r2 is referred to as the interest on interest. In our example, this latter amount is exactly: r2 5 ($. 09)2 5 $. 0081 When cash is invested at compound interest, each interest payment is reinvested. With simple interest, the interest is not reinvested. Benjamin Franklin’s statement, “ Money makes money and the money that money makes makes more money, " is a colorful way of explaining compound interest. The difference between compound interest and simple interest is illustrated in Figure 4. 4. In this example, the difference does not amount to much because the loan is for $1. If the loan were for $1 million, the lender would receive $1, 188, 100 in two years’ time. Of this amount, $8, 100 is interest on interest. The lesson is that those small numbers beyond the decimal point can add up to big dollar amounts when the transactions are for big amounts. In addition, the longer-lasting the loan, the more important interest on interest becomes. 92 Part II Valuation and Capital Budgeting Figure 4. 4 Simple and Compound Interest $1. 295 $1. 270 $1. 188 $1. 180 $1. 09 $1 1 year 2 years 3 years The dark-shaded area indicates the difference between compound and simple interest. The difference is substantial over a period of many years or decades. The general formula for an investment over many periods can be written as follows: Future Value of an Investment: FV 5 C0 3 (1 1 r)T (4. 3) where C0 is the cash to be invested at Date 0 (i. e., today), r is the interest rate per period, and T is the number of periods over which the cash is invested. EXAMPLE 4. 3 Interest on Interest Suh-Pyng Ku has put $500 in a savings account at the First National Bank of Kent. The account earns 7 percent, compounded annually. How much will Ms. Ku have at the end of three years? The answer is: $500 3 1. 07 3 1. 07 3 1. 07 5 $500 3 (1. 07)3 5 $612. 52 Figure 4. 5 illustrates the growth of Ms.

Ku’s account. Figure 4. 5 Suh-Pyng Ku’s Savings Account $612. 52 Dollars $500 $612. 52 0 1 2 Time 3 0 2$500 1 Time 2 3 Chapter 4 Discounted Cash Flow Valuation 93 EXAMPLE 4. 4 Compound Growth Jay Ritter invested $1, 000 in the stock of the SDH Company. The company pays a current dividend of $2, which is expected to grow by 20 percent per year for the next two years. What will the dividend of the SDH Company be after two years? A simple calculation gives: $2 3 (1. 20)2 5 $2. 88 Figure 4. 6 illustrates the increasing value of SDH’s dividends. Figure 4. 6 The Growth of the SDH Dividends $2. 88 Dollars $2. 40 $2. 00 Cash inï¬, ows $2. 40 $2. 00 $2. 88 0 1 Time 2 0 1 Time 2 The two previous examples can be calculated in any one of several ways. The computations could be done by hand, by calculator, by spreadsheet, or with the help of a table. We will introduce spreadsheets in a few pages, and we show how to use a calculator in Appendix 4B on the website. The appropriate table is Table A. 3, which appears in the back of the text. This table presents future value of $1 at the end of T periods. The table is used by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Suh-Pyng Ku would look at the following portion of Table A. 3: Interest Rate Period 1 2 3 4 6% 1. 0600 1. 1236 1. 1910 1. 2625 7% 1. 0700 1. 1449 1. 2250 1. 3108 8% 1. 0800 1. 1664 1. 2597 1. 3605 She could calculate the future value of her $500 as: $500 1. 2250 3 5 $612. 50 Initial Future value investment of $1 In the example concerning Suh-Pyng Ku, we gave you both the initial investment and the interest rate and then asked you to calculate the future value. Alternatively, the interest rate could have been unknown, as shown in the following example. 94 Part II Valuation and Capital Budgeting EXAMPLE 4. 5 Finding the Rate Carl Voigt, who recently won $10, 000 in the lottery, wants to buy a car in five years. Carl estimates that the car will cost $16, 105 at that time. His cash flows are displayed in Figure 4. 7. What interest rate must he earn to be able to afford the car? Figure 4. 7 Cash Flows for Purchase of Carl Voigt’s Car Cash inï¬, ow $10, 000 5 0 Cash outï¬, ow Time 2$16, 105 The ratio of purchase price to initial cash is: $16, 105 _______ $10, 000 5 1. 6105 Thus, he must earn an interest rate that allows $1 to become $1. 6105 in five years. Table A. 3 tells us that an interest rate of 10 percent will allow him to purchase the car. We can express the problem algebraically as: $10, 000 3 (1 1 r)5 5 $16, 105 where r is the interest rate needed to purchase the car. Because $16, 105/$10, 000 5 1. 6105, we have: (1 1 r)5 5 1. 6105 r 5 10% Either the table, a spreadsheet, or a hand calculator lets us solve for r. THE POWER OF COMPOUNDING: A DIGRESSION Most people who have had any experience with compounding are impressed with its power over long periods. Take the stock market, for example. Ibbotson and Sinquefield have calculated what the stock market returned as a whole from 1926 through 2010. 1 They find that one dollar placed in large U. S. stocks at the beginning of 1926 would have been worth $2, 982. 24 at the end of 2010. This is 9. 87 percent compounded annually for 85 years–that is, (1. 0987)85 5 $2, 982. 24, ignoring a small rounding error. The example illustrates the great difference between compound and simple interest. At 9. 87 percent, simple interest on $1 is 9. 87 cents a year. Simple interest over 85 years is $8. 39 (585 3 $. 0987). That is, an individual withdrawing 9. 87 cents every year would have withdrawn $8. 39 (585 3 $. 0987) over 85 years. This is quite a bit below the $2, 982. 24 that was obtained by reinvestment of all principal and interest. The results are more impressive over even longer periods. A person with no experience in compounding might think that the value of $1 at the end of 170 years would 1 Stocks, Bonds, Bills, and Inflation [SBBI]. 2011 Yearbook. Morningstar, Chicago, 2011. Chapter 4 Discounted Cash Flow Valuation 95 be twice the value of $1 at the end of 85 years, if the yearly rate of return stayed the same. Actually the value of $1 at the end of 170 years would be the square of the value of $1 at the end of 85 years. That is, if the annual rate of return remained the same, a $1 investment in common stocks would be worth $8, 893, 755. 42 (5$1 3 2, 982. 24 3 2, 982. 24). A few years ago, an archaeologist unearthed a relic stating that Julius Caesar lent the Roman equivalent of one penny to someone. Because there was no record of the penny ever being repaid, the archaeologist wondered what the interest and principal would be if a descendant of Caesar tried to collect from a descendant of the borrower in the 20th century. The archaeologist felt that a rate of 6 percent might be appropriate. To his surprise, the principal and interest due after more than 2, 000 years was vastly greater than the entire wealth on earth. The power of compounding can explain why the parents of well-to-do families frequently bequeath wealth to their grandchildren rather than to their children. That is, they skip a generation. The parents would rather make the grandchildren very rich than make the children moderately rich. We have found that in these families the grandchildren have a more positive view of the power of compounding than do the children. EXAMPLE 4. 6 How Much for That Island? Some people have said that it was the best real estate deal in history. Peter Minuit, director general of New Netherlands, the Dutch West India Company’s colony in North America, in 1626 allegedly bought Manhattan Island for 60 guilders’ worth of trinkets from native Americans. By 1667, the Dutch were forced by the British to exchange it for Suriname (perhaps the worst real estate deal ever). This sounds cheap; but did the Dutch really get the better end of the deal? It is reported that 60 guilders was worth about $24 at the prevailing exchange rate. If the native Americans had sold the trinkets at a fair market value and invested the $24 at 5 percent (tax free), it would now, about 385 years later, be worth more than $3. 45 billion. Today, Manhattan is undoubtedly worth more than $3. 45 billion, so at a 5 percent rate of return the native Americans got the worst of the deal. However, if invested at 10 percent, the amount of money they received would be worth about: $24(1 1 r)T 5 24 3 1. 1385 â‰… $207 quadrillion This is a lot of money. In fact, $207 quadrillion is more than all the real estate in the world is worth today. Note that no one in the history of the world has ever been able to find an investment yielding 10 percent every year for 385 years. PRESENT VALUE AND DISCOUNTING We now know that an annual interest rate of 9 percent enables the investor to transform $1 today into $1. 1881 two years from now. In addition, we would like to know the following: How much would an investor need to lend today so that she could receive $1 two years from today? Algebraically, we can write this as: PV 3 (1. 09)2 5 $1 96 Part II Valuation and Capital Budgeting Figure 4. 8 Compounding and Discounting Compounding at 9% Dollars $1, 000 $2, 367. 36 Compound interest $1, 900 Simple interest $1, 000 $422. 41 Discounting at 9% 1 2 3 4 5 6 7 8 9 10 Future years The top line shows the growth of $1, 000 at compound interest with the funds invested at 9 percent: $1, 000 3 (1. 09)10 5 $2, 367. 36. Simple interest is shown on the next line. It is $1, 000 1 [10 3 ($1, 000 3 . 09)] 5 $1, 900. The bottom line shows the discounted value of $1, 000 if the interest rate is 9 percent. In the preceding equation, PV stands for present value, the amount of money we must lend today to receive $1 in two years' time. Solving for PV in this equation, we have: $1 PV 5 ______ 5 $. 84 1. 1881 This process of calculating the present value of a future cash flow is called discounting. It is the opposite of compounding. The difference between compounding and discounting is illustrated in Figure 4. 8. To be certain that $. 84 is in fact the present value of $1 to be received in two years, we must check whether or not, if we lent $. 84 today and rolled over the loan for two years, we would get exactly $1 back. If this were the case, the capital markets would be saying that $1 received in two years' time is equivalent to having $. 84 today. Checking the exact numbers, we get: $. 84168 3 1. 09 3 1. 09 5 $1 In other words, when we have capital markets with a sure interest rate of 9 percent, we are indifferent between receiving $. 84 today or $1 in two years. We have no reason to treat these two choices differently from each other because if we had $. 84 today and lent it out for two years, it would return $1 to us at the end of that time. The value . 84 [51/(1. 09)2] is called the present value factor. It is the factor used to calculate the present value of a future cash flow. In the multiperiod case, the formula for PV can be written as follows: Present Value of Investment: CT PV 5 ________ (1 1 r)T Here, CT is the cash flow at date T and r is the appropriate discount rate. (4. 4) Chapter 4 Discounted Cash Flow Valuation 97 EXAMPLE 4. 7 Multiperiod Discounting Bernard Dumas will receive $10, 000 three years from now. Bernard can earn 8 percent on his investments, so the appropriate discount rate is 8 percent. What is the present value of his future cash flow? The answer is: 1 PV 5 $10, 000 3 ____ 1. 08 ( ) 3 5 $10, 000 3 . 7938 5 $7, 938 Figure 4. 9 illustrates the application of the present value factor to Bernard's investment. Figure 4. 9 Discounting Bernard Dumas's Opportunity $10, 000 Dollars $7, 938 Cash inï¬, ows 0 0 1 Time 2 3 1 2 Time

$10, 000 3 When his investments grow at an 8 percent rate of interest, Bernard Dumas is equally inclined toward receiving $7, 938 now and receiving $10, 000 in three years' time. After all, he could convert the $7, 938 he receives today into $10, 000 in three years by lending it at an interest rate of 8 percent. Bernard Dumas could have reached his present value calculation in one of several ways. The computation could have been done by hand, by calculator, with a spreadsheet, or with the help of Table A. 1, which appears in the back of the text. This table presents the present value of $1 to be received after T periods. We use the table by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Bernard Dumas would look at the following portion of Table A. 1: Interest Rate Period 1 2 3 4 7% . 9346 . 8734 . 8163 . 7629 8% . 9259 . 8573 . 7938 . 7350 9% . 9174 . 8417 . 7722 . 7084 The appropriate present value factor is . 7938. In the preceding example we gave both the interest rate and the future cash flow. Alternatively, the interest rate could have been unknown. 98 Part II Valuation and Capital Budgeting EXAMPLE 4. 8 Finding the Rate A customer of the Chaffkin Corp. wants to buy a tugboat today. Rather than paying immediately, he will pay $50, 000 in three years. It will cost the Chaffkin Corp. $38, 610 to build the tugboat immediately. The relevant cash flows to Chaffkin Corp. are displayed in Figure 4. 10. What interest rate would the Chaffkin Corp. charge to neither gain nor lose on the sale? Figure 4. 10 Cash Flows for Tugboat Cash inï¬, ows $50, 000 Time 0 Cash outï¬, ows 2$38, 610 3 The ratio of construction cost (present value) to sale price (future value) is: $38, 610 ________ $50, 000 5 . 7722 We must determine the interest rate that allows $1 to be received in three years to have a present value of

$. 7722. Table A. 1 tells us that 9 percent is that interest rate. FINDING THE NUMBER OF PERIODS Suppose we are interested in purchasing an asset that costs $50, 000. We currently have $25, 000. If we can earn 12 percent on this $25, 000, how long until we have the $50, 000? Finding the answer involves solving for the last variable in the basic present value equation, the number of periods. You already know how to get an approximate answer to this particular problem. Notice that we need to double our money. From the Rule of 72 (see Problem 74 at the end of the chapter), this will take about 72/12 5 6 years at 12 percent. To come up with the exact answer, we can again manipulate the basic present value equation. The present value is $25, 000, and the future value is $50, 000. With a 12 percent discount rate, the basic equation takes one of the following forms: $25, 000 5 $50, 000y1. 12t $50, 000y25, 000 5 1. 12t 5 2 We thus have a future value factor of 2 for a 12 percent rate. We now need to solve for t. If you look down the column in Table A. 3 that corresponds to 12 percent, you will see that a future value factor of 1. 9738 occurs at six periods. It will thus take about six years, as we calculated. To get the exact answer, we have to explicitly solve for t (by using a financial calculator or the spreadsheet on the next page). If you do this, you will see that the answer is 6. 1163 years, so our approximation was quite close in this case. Chapter 4 Discounted Cash Flow Valuation 99 SPREADSHEET APPLICATIONS Using a Spreadsheet for Time Value of Money Calculations More and more, businesspeople from many different areas (not just finance and accounting) rely on spreadsheets to do all the different types of calculations that come up in the real world. As a result, in this section, we will show you how to use a spreadsheet to handle the various time value of money problems we present in this chapter. We will use

Microsoft Excel™, but the commands are similar for other types of software. We assume you are already familiar with basic spreadsheet operations. As we have seen, you can solve for any one of To Find Enter This Formula the following four potential unknowns: future value, present value, the discount rate, or the number of peFuture value 5 FV (rate, nper, pmt, pv) riods. With a spreadsheet, there is a separate formula Present value 5 PV (rate, nper, pmt, fv) for each. In Excel, these are shown in a nearby box. In these formulas, pv and fv are present and fuDiscount rate 5 RATE (nper, pmt, pv, fv) ture value, nper is the number of periods, and rate Number of periods 5 NPER (rate, pmt, pv, fv) is the discount, or interest, rate. Two things are a little tricky here. First, unlike a financial calculator, the spreadsheet requires that the rate be entered as a decimal. Second, as with most financial calculators, you have to put a negative sign on either the present value or the future value to solve for the rate or the number of periods. For the same reason, if you solve for a present value, the answer will have a negative sign unless you input a negative future value. The same is true when you compute a future value. To illustrate how you might use these formulas, we will go back to an example in the chapter. If you invest $25, 000 at 12 percent per year, how long until you have $50, 000? You might set up a spreadsheet like this: A B C D E F G 1 Using a spreadsheet for time value of money calculations 2 3 4 If we invest $25, 000 at 12 percent, how long until we have $50, 000? We need to solve 5 for the unknown number of periods, so we use the formula NPER(rate, pmt, pv, fv). 6 7 Present value (pv): $25, 000 8 Future value (fv): $50, 000 9 Rate (rate): . 12 10 11 Periods: 6. 1162554 12 1 3 The formula entered in cell B11 is = NPER(B9, 0,-B7, B8); notice that pmt is zero and that pv 1 4 has a negative sign on it. Also notice that rate is entered as a decimal, not a percentage. H EXAMPLE 4. 9 Waiting for Godot You've been saving up to buy the Godot Company. The total cost will be $10 million. You currently have about $2. 3 million. If you can earn 5 percent on your money, how long will you have to wait? At 16 percent, how long must you wait? At 5 percent, you'll have to wait a long time. From the basic present value equation: $2. 3 million 5 $10 milliony1. 05t 1. 05t 5 4. 35 t 5 30 years At 16 percent, things are a little better. Verify for yourself that it will take about 10 years. 100 Learn more about using Excel for time value and other calculations at www. studyfinance. com. Part II Valuation and Capital Budgeting Frequently, an investor or a business will receive more than one cash flow. The present value of a set of cash flows is simply the sum of the present values of the individual cash flows. This is illustrated in the following two examples. EXAMPLE 4. 10 Cash Flow Valuation Kyle Mayer has won the Kentucky State Lottery and will receive the following set of cash flows over the next two years: Year 1 2 Cash Flow $20, 000 50, 000 Mr. Mayer can currently earn 6 percent in his money market account, so the appropriate discount rate is 6 percent. The present value of the cash flows is: Year 1 2 Cash Flow 3 Present Value Factor 5 Present Value 1 1 $20, 000 3 ____ 5 $20, 000 3 ____ 5 $18, 867. 9 1. 06 1. 06 1 2 1 $50, 000 3 ____ 5 $50, 000 3 ______ 5 $44, 499. 8 1. 06 (1. 06)2 Total $63, 367. 7 ( ) In other words, Mr. Mayer is equally inclined toward receiving $63, 367. 7 today and receiving $20, 000 and $50, 000 over the next two years. EXAMPLE 4. 11 NPV Finance. com has an opportunity to invest in a new high-speed computer that costs $50, 000. The computer will generate cash flows (from cost savings) of $25, 000 one year from now, $20, 000 two years from now, and $15, 000 three years from now. The computer will be worthless after three years, and no additional cash flows will occur. Finance. com has determined that the appropriate discount rate is 7 percent for this investment. Should Finance. com make this investment in a new high-speed computer? What is the net present value of the investment? The cash flows and present value factors of the proposed computer are as follows: Cash Flows Year 0 1 2 3 2$50, 000 $25, 000 $20, 000 $15, 000 Present Value Factor 151 1. 07 5 . 9346 1 ____ 2 1. 07 5 . 8734 3 1 ____ ( ) 1 (____) 5 . 8163 1. 07 Chapter 4 Discounted Cash Flow Valuation 101 The present value of the cash flows is: Cash Flows 3 Present value factor 5 Present value Year 0 1 2 3 2$50, 000 3 1 $25, 000 3 . 9346 $20, 000 3 . 8734 $15, 000 3 . 8163 5 5 5 5 Total: 2$50, 000 $23, 365 $17, 468 $12, 244. 5 $ 3, 077. 5 Finance. com should invest in the new high-speed computer because the present value of its future cash flows is greater than its cost. The NPV is $3, 077. 5. THE ALGEBRAIC FORMULA To derive an algebraic formula for the net present value of a cash flow, recall that the PV of receiving a cash flow one year from now is: PV 5 C1/(1 1 r) and the PV of receiving a cash flow two years from now is: PV 5 C2/(1 1 r)2 We can write the NPV of a T-period project as: CT C1 C2 NPV 5 âˆ’C0 1 _____ 1 _______ 1 . . . 1 ________ 5 âˆ’C0 1 1 1 r (1 1 r)2 (1 1 r)T âˆ‘ _______ (1 1 r) i i51 T Ci (4. 5) The initial flow, —C0, is assumed to be negative because it represents an investment. The o is shorthand for the sum of the series. We will close this section by answering the question we posed at the beginning of the chapter concerning baseball player Carl Crawford’s contract. Recall that the contract called for a $6 million signing bonus and $14 million in 2011. The remaining $122 million was to be paid as $19. 5 million in 2012, $20 million in 2013, $20. 25 million in 2014, $20. 5 million in 2015, $20. 75 million in 2016, and $21 million in 2017. If 12 percent is the appropriate interest rate, what kind of deal did the Red Sox outfielder snag? To answer, we can calculate the present value by discounting each year's salary back to the present as follows (notice we assume that all the payments are made at year-end): Year 0 (2010): $ 6, 000, 000 5 $ 6, 000, 000. 00 Year 1 (2011): $14, 000, 000 3 1/1. 121 5 $12, 500, 000. 00 Year 2 (2012): $19, 500, 000 3 1/1. 122 5 $15, 545, 280. 61 Year 3 (2013): $20, 000, 000 3 1/1. 123 5 $14, 235, 604. 96 . . . . Year 7 (2017): $21, 000, 000 3 1/1. 127 5 $ 9, 499, 332. 52 If you fill in the missing rows and then add (do it for practice), you will see that Carl's contract had a present value of about $92. 8 million, or only about 65 percent of the stated $142 million value (but still pretty good). 102 Part II Valuation and Capital Budgeting 4. 3 Compounding Periods So far, we have assumed that compounding and discounting occur yearly. Sometimes, compounding may occur more frequently than just once a year. For example, imagine that a bank pays a 10 percent interest rate " compounded semiannually. " This means that a $1, 000 deposit in the bank would be worth $1, 000 3 1. 05 5 $1, 050 after six months, and $1, 050 3 1. 05 5 $1, 102. 50 at the end of the year. The end-of-the-year wealth can be written as: . 10 2 $1, 000 1 + _ = $1, 000 Ã— (1. 05)2 = $1, 102. 50 2 Of course, a $1, 000 deposit would be worth $1, 100 (5$1, 000 3 1. 10) with yearly compounding. Note that the future value at the end of one year is greater with semiannual compounding than with yearly compounding. With yearly compounding, the original $1, 000 remains the investment base for the full year. The original $1, 000 is the investment base only for the first six months with semiannual compounding. The base over the second six months is $1, 050. Hence one gets interest on interest with semiannual compounding. Because $1, 000 3 1. 1025 5 $1, 102. 50, 10 percent compounded semiannually is the same as 10. 25 percent compounded annually. In other words, a rational investor could not care less whether she is quoted a rate of 10 percent compounded semiannually or a rate of 10. 25 percent compounded annually. Quarterly compounding at 10 percent yields wealth at the end of one year of: . 10 4 $1, 000 1 + _ = $1, 103. 81 4 More generally, compounding an investment m times a year provides end-of-year wealth of: r m C0(1 + _) (4. 6) m where C0 is the initial investment and r is the stated annual interest rate. The stated annual interest rate is the annual interest rate without consideration of compounding. Banks and other financial institutions may use other names for the stated annual interest rate. Annual percentage rate (APR) is perhaps the most common synonym. ( ) ( ) EXAMPLE 4. 12 EARs What is the end-of-year wealth if Jane Christine receives a stated annual interest rate of 24 percent compounded monthly on a $1 investment? Using Equation 4. 6, her wealth is: . 24 $1 1 + _ 12 ( ) 12 = $1 Ã — (1. 02)12 = $1. 2682 The annual rate of return is 26. 82 percent. This annual rate of return is called either the effective annual rate (EAR) or the effective annual yield (EAY). Due to compounding, the effective annual interest rate is greater than the stated annual interest rate of 24 percent. Algebraically, we can rewrite the effective annual interest rate as follows: Effective Annual Rate: r m (1 + _) âˆ’ 1 m (4. 7) Students are often bothered by the subtraction of 1 in Equation 4. 7. Note that end-of-year wealth is composed of both the interest earned over the year and the original principal. We remove the original principal by subtracting 1 in Equation 4. 7. Chapter 4 Discounted Cash Flow Valuation 103 EXAMPLE 4. 13 Compounding Frequencies If the stated annual rate of interest, 8 percent, is compounded quarterly, what is the effective annual rate? Using Equation 4. 7, we have: .

08 4 r m (1 + _) âˆ’ 1 = 1 + _ âˆ’ 1 = . 0824 = 8. 24% m (4) Referring back to our original example where C0 5 $1, 000 and r 5 10%, we can generate the following table: Effective Annual Rate m 5 r (1 1 __) 2 1 m . 10 . 1025 . 10381 . 10516 C0 $1, 000 1, 000 1, 000 1, 000 Compounding Frequency (m) Yearly (m = 1) Semiannually (m = 2) Quarterly (m = 4) Daily (m = 365) C1 $1, 100. 00 1, 102. 50 1, 103. 81 1, 105. 16 DISTINCTION BETWEEN STATED ANNUAL INTEREST RATE AND EFFECTIVE ANNUAL RATE The distinction between the stated annual interest rate (SAIR), or APR, and the effective annual rate (EAR) is frequently troubling to students. We can reduce the confusion by noting that the SAIR becomes meaningful only if the compounding interval is given. For example, for an SAIR of 10 percent, the future value at the end of one year with semiannual compounding is [1 + (. 10/2)]2 = 1. 1025. The future value with quarterly compounding is [1 + (. 10/4)]4 = 1. 1038. If the SAIR is 10 percent but no compounding interval is given, we cannot calculate future value. In other words, we do not know whether to compound semiannually, quarterly, or over some other interval. By contrast, the EAR is meaningful without a compounding interval. For example, an EAR of 10. 25 percent means that a $1 investment will be worth $1. 1025 in one year. We can think of this as an SAIR of 10 percent with semiannual compounding or an SAIR of 10. 25 percent with annual compounding, or some other possibility. There can be a big difference between an SAIR and an EAR when interest rates are large. For example, consider “ payday loans. " Payday loans are short-term loans made to consumers, often for less than two weeks. They are offered by companies such as Check Into Cash and AmeriCash Advance. The loans work like this: You write a check today that is postdated. When the check date arrives, you

go to the store and pay the cash for the check, or the company cashes the check. For example, in one particular state, Check Into Cash allows you to write a check for $115 dated 14 days in the future, for which they give you $100 today. So what are the APR and EAR of this arrangement? First, we need to find the interest rate, which we can find by the FV equation as follows: FV 5 PV 3 (1 1 r)1 $115 5 $100 3 (1 1 r)1 1. 15 5 (1 1 r) r 5 . 15 or 15% 104 Part II Valuation and Capital Budgeting That doesn’t seem too bad until you remember this is the interest rate for 14 days! The APR of the loan is: APR 5 . 15 3 365/14 APR 5 3. 9107 or 391. 07% And the EAR for this loan is: EAR 5 (1 1 Quoted rate/m)m 2 1 EAR 5 (1 1 . 15)365/15 2 1 EAR 5 37. 2366 or 3, 723. 66% Now that’s an interest rate! Just to see what a difference a small difference in fees can make, AmeriCash Advance will make you write a check for $117. 50 for the same amount. Check for yourself that the APR of this arrangement is 456. 25 percent and the EAR is 6, 598. 65 percent. Still not a loan we would like to take out! By law, lenders are required to report the APR on all loans. In this text, we compute the APR as the interest rate per period multiplied by the number of periods in a year. According to federal law, the APR is a measure of the cost of consumer credit expressed as a yearly rate, and it includes interest and certain noninterest charges and fees. In practice, the APR can be much higher than the interest rate on the loan if the lender charges substantial fees that must be included in the federally mandated APR calculation. COMPOUNDING OVER MANY YEARS Equation 4. 6 applies for an investment over one year. For an investment over one or more (T) years, the formula becomes this: Future Value with Compounding: r mT FV = C0(1 + _) m (4. 8) EXAMPLE 4. 14 Multiyear Compounding Harry DeAngelo is investing $5, 000 at a stated annual interest rate of 12 percent per year, compounded quarterly, for five years. What is his wealth at the end of five years? Using Equation 4. 8, his wealth is: . 12 4Ã—5 $5, 000 Ã— 1 + _ = $5, 000 Ã— (1. 03)20 = $5, 000 Ã— 1. 8061 = $9, 030. 50 4 ( ) CONTINUOUS COMPOUNDING The previous discussion shows that we can compound much more frequently than once a year. We could compound semiannually, quarterly, monthly, daily, hourly, each minute, or even more often. The limiting case would be to compound every infinitesimal instant, which is commonly called continuous compounding. Surprisingly, banks and other financial institutions sometimes quote continuously compounded rates, which is why we study them. Chapter 4 Discounted Cash Flow Valuation 105 Though the idea of compounding this rapidly may boggle the mind, a simple formula is involved. With continuous compounding, the value at the end of T years is expressed as: C0 Ã— e rT (4. 9) where C0 is the initial investment, r is the stated annual interest rate, and T is the number of years over which the investment runs. The number e is a constant and is approximately equal to 2. 718. It is not an unknown like C0, r, and T. EXAMPLE 4. 15 Continuous Compounding Linda DeFond invested $1, 000 at a continuously compounded rate of 10 percent for one year. What is the value of her wealth at the end of one year? From Equation 4. 9 we have: $1, 000 Ã— e . 10 = $1, 000 Ã— 1. 1052 = $1, 105. 20 This number can easily be read from Table A. 5. We merely set r, the value on the horizontal dimension, to 10 percent and T, the value on the vertical dimension, to 1. For this problem the relevant portion of the table is shown here: Period (T) 1 2 3 Continuously Compounded Rate (r) 9% 1. 0942 1. 1972 1. 3100 10% 1. 1052 1. 2214 1. 3499 11% 1. 1163 1. 2461 1. 3910 Note that a continuously compounded rate of 10 percent is equivalent to an annually compounded rate of 10. 52 percent. In other words, Linda DeFond would not care whether her bank quoted a continuously compounded rate of 10 percent or a 10. 52 percent rate, compounded annually. EXAMPLE 4. 16 Continuous Compounding, Continued Linda DeFond's brother, Mark, invested $1, 000 at a continuously compounded rate of 10 percent for two years. The appropriate formula here is: $1, 000 Ã— e . 10Ã—2 = $1, 000 Ã— e . 20 = $1, 221. 40 Using the portion of the table of continuously compounded rates shown in the previous example, we find the value to be 1. 2214. Figure 4. 11 illustrates the relationship among annual, semiannual, and continuous compounding. Semiannual compounding gives rise to both a smoother curve and a higher ending value than does annual compounding. Continuous compounding has both the smoothest curve and the highest ending value of all. 106 Part II Valuation and Capital Budgeting Figure 4. 11 Annual, Semiannual, and Continuous Compounding 4 Dollars Dollars 3 2 1 0 1 2 3 Years 4 5 Interest earned 4 3 2 1 0 1 2 3 Years 4 5 Dollars Interest earned 4 3 2 1 0 1 2 3 Years 4 5 Interest earned Annual compounding Semiannual compounding Continuous compounding EXAMPLE 4. 17 Present Value with Continuous Compounding The Michigan State Lottery is going to pay you $100, 000 at the end of four years. If the annual continuously compounded rate of interest is 8 percent, what is the present value of this payment? 1 1 $100, 000 Ã— _ = $100, 000 Ã— __ = $72, 616. 37 1. 3771 e . 08Ã—4 4. 4 Simplifications The first part of this chapter has examined the concepts of future value and present value. Although these concepts allow us to answer a host of problems concerning the time value of money, the human effort involved can be excessive. For example, consider a bank calculating the present value of a 20-year monthly mortgage. This mortgage has 240 (520 3

12) payments, so a lot of time is needed to perform a conceptually simple task. Because many basic finance problems are potentially time-consuming, we search for simplifications in this section. We provide simplifying formulas for four classes of cash flow streams: - - - - Perpetuity. Growing perpetuity. Annuity. Growing annuity. PERPETUITY A perpetuity is a constant stream of cash flows without end. If you are thinking that perpetuities have no relevance to reality, it will surprise you that there is a well-known case of an unending cash flow stream: The British bonds called consols. An investor purchasing a consol is entitled to receive yearly interest from the British government forever. Chapter 4 Discounted Cash Flow Valuation 107 How can the price of a consol be determined? Consider a consol that pays a coupon of C dollars each year and will do so forever. Simply applying the PV formula gives us: C C C PV = _ + __ + __ + . . . 1 + r (1 + r)2 (1 + r)3 where the dots at the end of the formula stand for the infinite string of terms that continues the formula. Series like the preceding one are called geometric series. It is well known that even though they have an infinite number of terms, the whole series has a finite sum because each term is only a fraction of the preceding term. Before turning to our calculus books, though, it is worth going back to our original principles to see if a bit of financial intuition can help us find the PV. The present value of the consol is the present value of all of its future coupons. In other words, it is an amount of money that, if an investor had it today, would enable him to achieve the same pattern of expenditures that the consol and its coupons would. Suppose an investor wanted to spend exactly C dollars each year. If he had the consol, he could do this. How much money must he have today to spend the same amount? Clearly, he would need exactly enough so that the interest on the money

would be C dollars per year. If he had any more, he could spend more than C dollars each year. If he had any less, he would eventually run out of money spending C dollars per year. The amount that will give the investor C dollars each year, and therefore the present value of the consol, is simply: C (4. 10) PV = __ r To confirm that this is the right answer, notice that if we lend the amount Cyr, the interest it earns each year will be: C Interest = __ Ã— r = C r which is exactly the consol payment. We have arrived at this formula for a consol: Formula for Present Value of Perpetuity: C C C PV = _ + __ + __ + . . . (4. 11) 1 + r (1 + r)2 (1 + r)3 C = __ r It is comforting to know how easily we can use a bit of financial intuition to solve this mathematical problem. EXAMPLE 4. 18 Perpetuities Consider a perpetuity paying $100 a year. If the relevant interest rate is 8 percent, what is the value of the consol? Using Equation 4. 10 we have: $100 PV = _ = $1, 250 . 08 Now suppose that interest rates fall to 6 percent. Using Equation 4. 10 the value of the perpetuity is: $100 PV = _ = $1, 666. 67 . 06 Note that the value of the perpetuity rises with a drop in the interest rate. Conversely, the value of the perpetuity falls with a rise in the interest rate. 108 Part II Valuation and Capital Budgeting GROWING PERPETUITY Imagine an apartment building where cash flows to the landlord after expenses will be $100, 000 next year. These cash flows are expected to rise at 5 percent per year. If one assumes that this rise will continue indefinitely, the cash flow stream is termed a growing perpetuity. The relevant interest rate is 11 percent. Therefore, the appropriate discount rate is 11 percent, and the present value of the cash flows can be represented as: $100, 000 $100, 000(1. 05) $100, 000(1. 05)2 PV = __ + ___ + ___ + . . . 1. 11 (1. 11)2 (1. 11)3 Nâˆ’1 $100, 000(1. 05) + ____ + . . . (1. 11)N Algebraically, we can write the formula as: C Ã— (1 + g)

C Ã— (1 + g) C Ã— (1 + g) C PV = _ + ___ + ___ + . . . + ___ + . . . 1+r (1 + r)2 (1 + r)3 (1 + r)N 2 Nâˆ’1 where C is the cash flow to be received one period hence, g is the rate of growth per period, expressed as a percentage, and r is the appropriate discount rate. Fortunately, this formula reduces to the following simplification: Formula for Present Value of Growing Perpetuity: C PV = _ râˆ’g $100, 000 __ = $1, 666, 667 . 11 âˆ’ . 05 There are three important points concerning the growing perpetuity formula: 1. The numerator: The numerator in Equation 4. 12 is the cash flow one period hence, not at date 0. Consider the following example. (4. 12) From Equation 4. 12 the present value of the cash flows from the apartment building is: EXAMPLE 4. 19 Paying Dividends Popovich Corporation is just about to pay a dividend of $3. 00 per share. Investors anticipate that the annual dividend will rise by 6 percent a year forever. The applicable discount rate is 11 percent. What is the price of the stock today? The numerator in Equation 4. 12 is the cash flow to be received next period. Since the growth rate is 6 percent, the dividend next year is $3. 18 (5$3. 00 3 1. 06). The price of the stock today is: $66. 60 = $3. 00 Imminent dividend + $3. 18 __ . 11 âˆ’ . 06 Present value of all dividends beginning a year from now The price of $66. 60 includes both the dividend to be received immediately and the present value of all dividends beginning a year from now. Equation 4. 12 makes it possible to calculate only the present value of all dividends beginning a year from now. Be sure you understand this example; test questions on this subject always seem to trip up a few of our students. 2. The discount rate and the growth rate: The discount rate r must be greater than the growth rate g for the growing perpetuity formula to work. Consider the case Chapter 4 Discounted Cash Flow Valuation 109 3. in which the growth rate approaches the interest rate in magnitude. Then, the denominator in the growing perpetuity formula gets infinitesimally small and the present value grows infinitely large. The present value is in fact undefined when r is less than g. The timing assumption: Cash generally flows into and out of real-world firms both randomly and nearly continuously. However, Equation 4. 12 assumes that cash flows are received and disbursed at regular and discrete points in time. In the example of the apartment, we assumed that the net cash flows of $100, 000 occurred only once a year. In reality, rent checks are commonly received every month. Payments for maintenance and other expenses may occur anytime within the year. We can apply the growing perpetuity formula of Equation 4. 12 only by assuming a regular and discrete pattern of cash flow. Although this assumption is sensible because the formula saves so much time, the user should never forget that it is an assumption. This point will be mentioned again in the chapters ahead. A few words should be said about terminology. Authors of financial textbooks generally use one of two conventions to refer to time. A minority of financial writers treat cash flows as being received on exact dates–for example Date 0, Date 1, and so forth. Under this convention, Date 0 represents the present time. However, because a year is an interval, not a specific moment in time, the great majority of authors refer to cash flows that occur at the end of a year (or alternatively, the end of a period). Under this end-of-the-year convention, the end of Year 0 is the present, the end of Year 1 occurs one period hence, and so on. (The beginning of Year 0 has already passed and is not generally referred to.)2 The interchangeability of the two conventions can be seen from the following chart: Date 0 = Now End of Year 0 = Now Date 1 End of Year 1 Date 2 End of Year 2 Date 3 End of Year 3 … … We strongly believe that the dates convention reduces ambiguity. However, we use both conventions because you are likely to see the end-of-year convention in later courses. In fact, both conventions may appear in the same example for the sake of practice. ANNUITY An annuity is a level stream of regular payments that lasts for a fixed number of periods. Not surprisingly, annuities are among the most common kinds of financial instruments. The pensions that people receive when they retire are often in the form of an annuity. Leases and mortgages are also often annuities. To figure out the present value of an annuity we need to evaluate the following equation: C C C C _ + __ + __ + . . . + __ 1 + r (1 + r)2 (1 + r)3 (1 + r)T 2 Sometimes, financial writers merely speak of a cash flow in Year x. Although this terminology is ambiguous, such writers generally mean the end of Year x. 110 Part II Valuation and Capital Budgeting The present value of receiving the coupons for only T periods must be less than the present value of a consol, but how much less? To answer this, we have to look at consols a bit more closely. Consider the following time chart: Now (T + 1) (T + 2) C C ... C C ... Date (or end of year) Consol 1 Consol 2 Annuity 0 1 C C 2 C C 3 C ... C ... T C C Consol 1 is a normal consol with its first payment at Date 1. The first payment of consol 2 occurs at Date T 1 1. The present value of having a cash flow of C at each of T dates is equal to the present value of Consol 1 minus the present value of Consol 2. The present value of Consol 1 is given by: C PV = _ (4. 13) r Consol 2 is just a consol with its first payment at Date T 1 1. From the perpetuity formula, this consol will be worth Cyr at Date T. 3 However, we do not want the value at date T. We want the value now, in other words, the present value at Date 0. We must discount Cyr back by T periods. Therefore, the present value of Consol 2 is: C 1 PV = _ __ r (1 + r)T [ ] (4. 14) The present value of having cash flows for T years is the present value of a consol with its first payment at Date 1 minus the present value of a consol with its first payment at Date T 1 1. Thus the present value of an annuity is Equation 4. 13 minus Equation 4. 14. This can be written as: C C 1 _ âˆ’ _ __ r r (1 + r)T This simplifies to the following: Formula for Present Value of Annuity: 1 1 PV = C _ âˆ’ __ r r(1 + r)T [ ]] [This can also be written as: PV = C As per CE MS, it is fine? 3 [1 1 2 _______ (1 + r)T ___________ r] (4. 15) Students frequently think that C/r is the present value at Date T 1 1 because the consol’s first payment is at Date T 1 1. However, the formula values the consol as of one period prior to the first payment. Chapter 4 Discounted Cash Flow Valuation 111 EXAMPLE 4. 20 Lottery Valuation Mark Young has just won the state lottery, paying $50, 000 a year for 20 years. He is to receive his first payment a year from now. The state advertises this as the Million Dollar Lottery because $1, 000, 000 5 $50, 000 3 20. If the interest rate is 8 percent, what is the present value of the lottery? Equation 4. 15 yields: 1 1 âˆ’ __ (1. 08)20 ___ = $50, 000 Ã— . 08 Present value of Million Dollar Lottery [ ] Periodic payment Annuity factor = $50, 000 Ã— 9. 8181 = $490, 905 Rather than being overjoyed at winning, Mr. Young sues the state for misrepresentation and fraud. His legal brief states that he was promised $1 million but received only $490, 905. The term we use to compute the present value of the stream of level payments, C, for T years is called a present value interest factor in annuities. The present value interest factor in annuities in the current example is 9. 8181. Because the present value interest factor in annuities is used so often in PV calculations, we have included it in Table A. 2 in the back of this book. The table gives the values of these factors for a range of interest rates, r, and maturity dates, T. The present value interest factor in annuities as expressed in the brackets of Equation 4. 15 is a complex formula. For simplification, we may from time to time refer to the annuity factor as: PVIA (r, T) This expression stands for the present value of $1 a year for T years at an interest rate of r. We can also provide a formula for the future value of an annuity: (1 + r)T 1 (1 + r)T âˆ’ 1 FV = C __ âˆ’ _ = C ___ r r r [ ] [ ] (4. 16) As with present value factors for annuities, we have compiled future value factors in Table A. 4 in the back of this book. EXAMPLE 4. 21 Retirement Investing Suppose you put $3, 000 per year into a Roth IRA. The account pays 6 percent interest per year. How much will you have when you retire in 30 years? This question asks for the future value of an annuity of $3, 000 per year for 30 years at 6 percent, which we can calculate as follows: (1 + r)T âˆ’ 1 1. 0630 âˆ’ 1 FV = C ___ = $3, 000 Ã— __ r . 06 = $3, 000 Ã— 79. 0582 = $237, 174. 56 [ ] [ ] So, you'll have close to a quarter million dollars in the account. Our experience is that annuity formulas are not hard, but tricky, for the beginning student. We present four tricks next. 112 Part II Valuation and Capital Budgeting SPREADSHEET APPLICATIONS Annuity Present Values Using a spreadsheet to find annuity present values goes like this: A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 B C D E F G Using a spreadsheet to find annuity present values What is the present value of $500 per year for 3 years if the discount rate is 10 percent? We need to solve for the unknown present value, so we use the formula PV(rate, nper, pmt, fv). Payment amount per period: Number of payments: Discount rate: Annuity present value: $500 3 0. 1 $1, 243. 43 The formula entered in cell B11 is = PV(B9, B8,-B7, 0); notice that fv is zero and that pmt has a negative sign on it. Also notice that rate is entered as a decimal, not a percentage. Trick 1: A Delayed Annuity One of the tricks in working with annuities or perpetuities is getting the timing exactly right. This is particularly true when an annuity or perpetuity begins at a date many periods in the future. We have found that even the brightest beginning student can make errors here. Consider the following example. EXAMPLE 4. 22 Delayed Annuities Danielle Caravello will receive a four-year annuity of $500 per year, beginning at date 6. If the interest rate is 10 percent, what is the present value of her annuity? This situation can be graphed as follows: 0 1 2 3 4 5 6 $500 7 $500 8 $500 9 $500 10 The analysis involves two steps: 1. Calculate the present value of the annuity using Equation 4. 15: Present Value of Annuity at Date 5: 1 1 âˆ’ __ (1. 10)4 __ = $500 Ã— PVIA (. 10, 4) $500 . 10 = $500 Ã— 3. 1699 = $1, 584. 95 Note that $1, 584. 95 represents the present value at Date 5. [ ] Chapter 4 Discounted Cash Flow Valuation 113 Students frequently think that $1, 584. 95 is the present value at Date 6 because the annuity begins at Date 6. However, our formula values the annuity as of one period prior to the first payment. This can be seen in the most typical case where the first payment occurs at Date 1. The formula values the annuity as of Date 0 in that case. 2. Discount the present value of the annuity back to Date 0: Present Value at Date 0: $1, 584. 95 __ = $984. 13 (1. 10)5 Again, it is worthwhile mentioning that because the annuity formula brings Danielle’s annuity back to Date 5, the second calculation must discount over the remaining five periods. The two-step procedure is graphed in Figure 4. 12. Figure 4. 12 Discounting Danielle Caravello’s Annuity Date 0 Cash ï¬, ow $984. 13 1 2 3 4 5 6 $500 7 $500 8 $500 9 $500 10 $1, 584. 95 Step one: Discount the four payments back to Date 5 by using the annuity formula. Step two: Discount the present value at Date 5 ($1, 584. 95) back to present value at Date 0. Trick 2: Annuity Due The annuity formula of Equation 4. 15 assumes that the first annuity payment begins a full period hence. This type of annuity is sometimes called an annuity in arrears or an ordinary annuity. What happens if the annuity begins today–in other words, at Date 0? EXAMPLE 4. 23 Annuity Due In a previous example, Mark Young received $50, 000 a year for 20 years from the state lottery. In that example, he was to receive the first payment a year from the winning date. Let us now assume that the first payment occurs immediately. The total number of payments remains 20. Under this new assumption, we have a 19-date annuity with the first payment occurring at Date 1–plus an extra payment at Date 0. The present value is: $50, 000 Payment at date 0 + $50, 000 Ã— PVIA (. 08, 19) 19-year annuity = $50, 000 + ($50, 000 Ã— 9. 6036) = $530, 180 $530, 180, the present value in this example, is greater than $490, 905, the present value in the earlier lottery example. This is to be expected because the annuity of the current example begins earlier. An annuity with an immediate initial payment is called an annuity in advance or, more commonly, an annuity due. Always remember that Equation 4. 15 and Table A. 2 in this book refer to an ordinary annuity. 114 Part II Valuation and Capital Budgeting Trick 3: The Infrequent Annuity The following example treats an annuity with payments occurring less frequently than once a year. EXAMPLE 4. 24 Infrequent Annuities Ann Chen receives an annuity of $450, payable once every two years. The annuity stretches out over 20 years. The first payment occurs at Date 2–that is, two years from today. The annual interest rate is 6 percent. The trick is to determine the interest rate over a two-year period. The interest rate over two years is: (1. 06 Ã— 1. 06) âˆ’ 1 = 12. 36% That is, $100 invested over two years will yield $112. 36. What we want is the present value of a $450 annuity over 10 periods, with an interest rate of 12. 36 percent per period: 1 1 â ˆ' ___ (1 + . 1236)10 ___ = $450 Ã— PVIA (. 1236, 10) = $2, 505. 57 $450 . 1236 [ ] Trick 4: Equating Present Value of Two Annuities The following example equates the present value of inflows with the present value of outflows. EXAMPLE 4. 25 Working with Annuities Harold and Helen Nash are saving for the college education of their newborn daughter, Susan. The Nashes estimate that college expenses will run $30, 000 per year when their daughter reaches college in 18 years. The annual interest rate over the next few decades will be 14 percent. How much money must they deposit in the bank each year so that their daughter will be completely supported through four years of college? To simplify the calculations, we assume that Susan is born today. Her parents will make the first of her four annual tuition payments on her 18th birthday. They will make equal bank deposits on each of her first 17 birthdays, but no deposit at Date 0. This is illustrated as follows: Date 0 1 2 ... 17 18 19 20 21 Susan's Parents' Parents' . . . Parents' Tuition Tuition Tuition Tuition birth 1st 2nd 17th and payment payment payment payment deposit deposit last 1 2 3 4 deposit Mr. and Ms. Nash will be making deposits to the bank over the next 17 years. They will be withdrawing $30, 000 per year over the following four years. We can be sure they will be able to withdraw fully $30, 000 per year if the present value of the deposits is equal to the present value of the four $30, 000 withdrawals. This calculation requires three steps. The first two determine the present value of the withdrawals. The final step determines yearly deposits that will have a present value equal to that of the withdrawals. Chapter 4 Discounted Cash Flow Valuation 115 1. We calculate the present value of the four years at college using the annuity formula: 1 1 â ˆ' __ (1. 14)4 __ = $30, 000 Ã— PVIA (. 14, 4) $30, 000 Ã— . 14 [ ] = $30, 000 Ã— 2. 9137 = $87, 411 We assume that Susan enters college on her 18th birthday. Given our discussion in Trick 1, $87, 411 represents the present value at Date 17. 2. We calculate the present value of the college education at Date 0 as: $87, 411 __ = $9, 422. 91 (1. 14)17 3. Assuming that Harold and Helen Nash make deposits to the bank at the end of each of the 17 years, we calculate the annual deposit that will yield a present value of all deposits of $9, 422. 91. This is calculated as: C Ã— PVIA (. 14, 17) = $9, 422. 91 Because PVIA (. 14, 17) = 6. 3729, $9, 422. 91 C = __ = $1, 478. 59 6. 3729 Thus deposits of $1, 478. 59 made at the end of each of the first 17 years and invested at 14 percent will provide enough money to make tuition payments of $30, 000 over the following four years. An alternative method in Example 4. 25 would be to (1) calculate the present value of the tuition payments at Susan's 18th birthday and (2) calculate annual deposits so that the future value of the deposits at her 18th birthda