Geometrical application of ordinary differential equation

Many practical problems in science and engineering are formulated by finding how one quantity is related to, or depends upon, one or more (other) quantities defined In the problem. Often, it is easier to model a relation between the rates of changes in the variable rather than between the variables themselves. This study of this relationship gives rise to differential equation. Derivatives can always be interpreted as rate. For example, if x is a function of t then dx/dt is the rate of x with respect to t. if x denotes the displacement of a particle, then dx/dt represents the velocity of the particle. If x represents the electric charge then dx/dt represents the flow of charge that is the current. Derivatives of higher orders represents rate of rates. If x denotes the displacement of particle, then d2x/dt2 represents the accelerations.

A differential equation can be defined as an equation containing derivatives of various orders and variables . differential equation which involves one independent variable are called ordinary differential equation. If the differential equation involves more than one independent variable and partial derivatives of the dependent variable with respect to them, than it is called partial differential equation.

Explanation:- Let y be the dependent variable and x be the independent variable. So the system can be denoted as

dy/dx= y’ , d2y/dx2= y”

Some Example of Ordinary Differential equation

y’= 6×2

$y'' + 16y = 2x$

$x2y'' - xy' + 6y = \log x$

$y'y'' + y2 = x2$

Introduction to differential equation, and solving linear differential equations using operator method:-

In this Term paper, I will first introduce what differential equation is? Separable first order differential equation will be solved. Then the integrating factor will be taught to solve linear differential equation of the first degree. The auxiliary equation (or characteristic equation) will be introduced to solve homogeneous linear equations, and then operator method will be taught finally to solve non-homogeneous linear equations.

This term paper assumes readers familiar with basic of calculus, like differentiation and integration.

What is differential equation?

A differential equation is an equation which contains derivatives. Here are some examples:

In these equations, y is an unknown function depends on x which we would like to solve. These kind of equations are very important in different fields, like in chemistry describing rate of reaction, physics describing equation of motion, etc. Therefore, able to solve these equations analytically enables us to understand many natural process. The above equations are known as ordinary differential equations(ODE) since they only contain derivatives with respect to one variable, x. (note that the equations hold for all values of x)

In mathematics, an ordinary differential equation (or ODE) is a relation that contains functions of only one independent variable, and one or more of their derivatives with respect to that variable.

A simple example is Newton’s second law of motion, which leads to the differential equation

for the motion of a particle of constant mass m. In general, the force F depends upon the position x(t) of the particle at time t, and thus the unknown function x(t) appears on both sides of the differential equation, as is indicated in the notation F(x(t)).

Ordinary differential equations are distinguished from partial differential equations, which involve partial derivatives of functions of several variables.

Ordinary differential equations arise in many different contexts including geometry, mechanics, astronomy and population modelling. Many famous mathematicians have studied differential equations and contributed to the field, including Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, d’Alembert and Euler.

Much study has been devoted to the solution of ordinary differential equations. In the case where the equation is linear, it can be solved by analytical methods. Unfortunately, most of the interesting differential equations are non-linear and, with a few exceptions, cannot be solved exactly. Approximate solutions are arrived at using computer approximations.

The trajectory of a projectile launched from a cannon follows a curve determined by an ordinary differential equation that is derived from Newton's second law.

Ordinary differential equation

Let y be an unknown function

in x with y(n) the nth derivative of y, and let F be a given function

then an equation of the form

is called an ordinary differential equation (ODE) of order n. If y is an unknown vector valued function

,

it is called a system of ordinary differential equations of dimension m (in this case, F : â„mn+1â†' â„m).

More generally, an implicit ordinary differential equation of order n has the form

where F : â„n+2â†' â„ depends on y(n). To distinguish the above case from this one, an equation of the form

is called an explicit differential equation.

A differential equation not depending on x is called autonomous.

A differential equation is said to be linear if F can be written as a linear combination of the derivatives of y together with a constant term, all possibly depending on x:

with ai(x) and r(x) continuous functions in x. The function r(x) is called the source term; if r(x)= 0 then the linear differential equation is called homogeneous, otherwise it is called non-homogeneous or inhomogeneous.

Solutions

Given a differential equation

a function u: I âŠ‚ R â†’ R is called the solution or integral curve for F, if u is n-times differentiable on I, and

Given two solutions u: J âŠ‚ R â†’ R and v: I âŠ‚ R â†’ R, u is called an extension of v if I âŠ‚ J and

A solution which has no extension is called a global solution.

Terminology

Partial differential equations

These are equations which involves more than one independent variable. For instance:

Partial differential equations(PDE) are significantly more difficult than ODE, and we won’t talk about it at this moment.

Order

Order of a differential equations is the order of the highest derivative in the equation.

Order 1:

Order 2:

Degree

The degree of a differential equation is the degree of the highest derivative in the equation.

Degree 1:

Degree 2:

Separable 1st order ODE

If the ODE is in the following form, the solution can be found using integration easily:

Example:-

In the study of partial differentiation, recall that a function of two variables that equals a constant, describes the points in the 3-D plane with the same potential; . The curves that connect the points with the same potential are called level curves and have the value of c. A contour map is a level curve graph where common elevations are connected giving a 2-D representation of a 3-D reality.

Using a LiveMath 3-D graph theory you can plot such a function along with the level curves describing the contours associated with that function. The picture below uses the following function to demonstrate this (here is a LiveMath plug-in animation of the graph below).

In general terms, this type of equation is represented by the following:

It describes the level curves and is the solution to the following differential equation. The equation below is just the total derivative of the function above.

Because it is the total derivative of some function z(x, y) it is called an Exact Differential Equation.

To help understand how to solve these types of equations you will look at the solution first and then analyze how to back into that solution.

In this example you will take the total derivative of a function and analyze its parts. Then you will take this new equation (a differential equation now) and, knowing the answer, describe the method used to solve it.

Input the following equation:

To take a total derivative in LiveMath, first input the differential operator d times z (d*z). Input this into a second Prop and substitute the equation into it.

Collect common terms on the RHS and Expand the coefficients of the differentials for the final answer.

After setting the RHS equal to zero you will have a differential equation to solve. Notice how the coefficients are neither separable, homogeneous, nor are they linear.

To help analyze this equation, label the coefficient of dx as M and the coefficient of dy as N. Both are functions of x and y so place the equation in the following form.

The total derivative of a function is obtained by adding the partial derivatives of the coefficients. This is done with the equation below.

Set up our notebook in the following manner:

Perform the substitutions to give the partial derivatives.

we can see that differential equations of this type are Exact. They are immediate derivatives of another function. You know this is true in this example because you developed the equation below by taking the derivative of the original function.

we can describe an Exact Differential Equation as an equation whose dx coefficient is the partial derivative with respect to x of some function f(x, y) and whose dy coefficient is the partial derivative with respect to y of the SAME function. Since you dry-labed the last example you know what this equation, z= f(x, y), is. This will not be the case as you look to solve these problems though, so you need to find a way of determining that an equation is exact, then you will know that M and N are related to the solution equation in this way!

Using the fact that these partial derivatives are of the same function will be the key to the method used to solve these equations. To test a differential equation for exactness, follow the method described in the next example.

Test for Exactness

This example demonstrates the test for exactness of the same equation used in the previous example. First input the differential equation as shown below. Remember to include an Independence Declaration inside the same case theory the test is performed.

To test for exactness, equate the partial derivative with respect to y of M and the partial derivative with respect to x of N. Notice that these partials are with respect to the exact opposite variables as those used to determine the total derivative in the last example. The reason for this will become clear to you later when you derive this test.

Set up the partial derivatives and solve by substituting M and N into the partial derivative Ops.

The fact that they are equal means that the differential equation is exact!

Method of Solution:

To solve these types of equations you will need to take one or the other coefficient and " go backwards" to determine the solution. You can take either M or N to do this, it is up to you. For this example try using M.

Solution Method

First, set up an equation equating the unknown function, named , to an integral of M plus some unknown function of y. Call this function u for the time being. The reason you do this is the fact that to get M, the partial derivative was taken of the unknown function with respect to x. You will try to back into the answer by integrating M. This is not automatic though, because of the fact that when a partial derivative is performed, one of the variables is treated as a constant and therefore drops out (the derivative of a constant is = 0). Below this derivative is displayed again.

We will not get back the function by integrating M, because the y term is not there! It is the constant, as is shown below where you try to get the function back by integrating M.

This is very close to the answer, and with a little twist, will lead to a method that you will use to obtain the solution.

Input the following props and perform the substitutions as shown. The user defined variable u is used in this case, rather than the arbitrary constant c, because you are actually looking for, what you might call, an arbitrary function. It will also be necessary later to have u defined as a variable for LiveMath to solve for the function.

Now we have a potential function (), that represents the solution to the problem. To solve, the function u must be determined. If you take the partial derivative of the unknown function with respect to y this time, you will get N. By setting up the equation this way, you can then isolate u.

You already know what N is, so:

Next substitute the potential function into the Prop and solve for u by performing an integration.

The final solution is achieved by substituting this u Prop back into the function . Since this function describes level curves, it is set equal to a constant c.

The question remains, why do you take partial derivatives of M and N to determine if an equation is exact? M and N have been defined as the partial derivatives of z with respect to x and y respectively.

By taking the second partial derivative of each coefficient WITH RESPECT TO THE OPPOSITE VARIABLE, the LHS of both of these equations is equal and therefore the RHS are equal too.