A remark on the fractional integral operators and the image formulas of generaliz...

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1. Introduction and Preliminaries

Fractional calculus (FC) represents a complex physical phenomenon in a more accurate and efficient way than classical calculus. In recent years, many researchers [1 – 7] have used fractional order integral models in real-world problems in various fields of science and technology. There exists several definitions of fractional order integrals in the literature that can be used to solve the fractional integral equations involving special functions. For an exhaustive literature review, one may refer to the paper by Srivastava and Saxena [8].

The generalized functions such as Bessel, Lommel, Struve, and Lommel–Wright functions have originated from concrete problems in applied fields of sciences viz mechanics, physics, engineering, astronomy, etc.

The *generalized Lommel–Wright function* $J_{\omega,\vartheta}^{\varphi,m}(z)$ is defined by de'Oteiza et al. [9] and is represented in the following manner:

$$J_{\omega,\vartheta}^{\varphi,m}(z) = (z/2)^{\omega+2\vartheta}\sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{2k}}{(\Gamma(\vartheta+k+1))^m\,\Gamma(\omega+k\varphi+\vartheta+1)} = (z/2)^{\omega+2\vartheta}\,{}_1\psi_{m+1}\left[(1,1);\underbrace{(\vartheta+1,1)}_{m-\text{times}},(\omega+\vartheta+1,\varphi);-z^2/4\right]$$

$$z\in\mathbb{C}(-\infty,0],\ \varphi>0,\ m\in\mathbb{N},\ \omega,\vartheta\in\mathbb{C}, \quad (1.1)$$

where ${}_p\psi_q$ denotes the Fox–Wright generalized hypergeometric function which is defined as given in Srivastava and Karlsson [10 , p. 21] and Kilbas et al. [11 , P. 56]

$${}_p\psi_q\left((a_1,A_1),\dots(a_p,A_p);(b_1,B_1),\dots,(b_q,B_q);z\right)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{z^n}{n!}, \quad (1.2)$$

where $a_i, b_j \in \mathbb{C}$ and $A_i, B_j \in \mathbb{R} = (-\infty, \infty)$; $A_i, B_j \neq 0$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$, $\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > -1$.

A useful generalization of the Lommel–Wright function and its special cases, $J_\omega^\varphi(z)$ $J_{\omega,\vartheta}^\varphi(z)$, depending on the arbitrary fractional parameter $\varphi > 0$ presents a fractional order extension of the Bessel function $J_\omega(z)$.

Prieto et al. [12] studied some useful results in the theory of fractional calculus operators of generalized Lommel–Wright function. The convergence of series involving generalized Lommel–Wright function was studied by Konovska [13].

When $m = 1$, the following generalization of the Bessel function, introduced by Pathak [14] is obtained as a special case of generalized Lommel–Wright function (1. 1) (see e. g., [15, p. 353]):

$$J_{\omega,\vartheta}^{\varphi}(z)=J_{\omega,\vartheta}^{\varphi,1}(z)=\left(\frac{z}{2}\right)^{\omega+2\vartheta}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{\Gamma(\vartheta+k+1)\,\Gamma(\omega+k\varphi+\vartheta+1)}, \quad z\in\mathbb{C}\setminus(-\infty,0],\ \varphi>0,\ \omega,\vartheta\in\mathbb{C}. \quad (1.3)$$

On taking $m = 1$, $\varphi = 1$, and $\vartheta = \frac{1}{2}$ in (1. 1), we obtain the *Struve function* $H_\omega(\cdot)$ (see e. g., [16, p. 28, Equation (1. 170)])

$$H_\omega(z)=J_{\omega,1/2}^{1,1}=\left(\frac{z}{2}\right)^{\omega+1}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\omega+\frac{3}{2}\right)}, \quad z,\omega\in\mathbb{C}. \quad (1.4)$$

If we take $m = 1$, $\varphi = 1$, and $\vartheta = 0$ in (1. 1), it gives the relationship with the *Bessel function* as follows (see e. g., [16 , p. 27, Equation (1. 161)]):

$$J_{\omega}(z) = J_{\omega,0}^{1,1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\omega+2k}}{\Gamma(\omega+k+1)\, k!}, \quad z, \omega \in \mathbb{C},\ z \neq 0,\ \Re(\omega) > -1. \quad (1.5)$$

A generalization of the hypergeometric fractional integrals, including the Saigo operators [17 , 18] has been introduced by Marichev [19]. The details of these fractional operators have been found in Samko et al. [5 , p. 194, Equation (10. 47)] and later extended and studied by Saigo and Maeda [20 , p. 393, Equation (4. 12) and Equation (4. 13)] in terms of complex order Appell function $F_3(\cdot)$ of two variables (see [10 , p. 23]) in the kernel

$$F_3(\zeta, \zeta', \varrho, \varrho'; \eta; x; y) = \sum_{m,n=0}^{\infty} \frac{(\zeta)_m (\zeta')_n (\varrho)_m (\varrho')_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (\max\{|x|, |y|\} < 1). \quad (1.6)$$

The Appell function F_3 reduces to the Gauss hypergeomatric function ${}_2F_1$ and satisfies the system of two linear partial differential equations of the second order as follows (see [10 , p. 301, Equation 9. 4]):

$$F_3(\zeta, \eta-\zeta, \varrho, \eta-\varrho; \eta; x; y) = {}_2F_1(\zeta, \varrho; \eta; x+y-xy). \quad (1.7)$$

Further, it is easy to see that

$$F_3(\zeta, 0, \varrho, \varrho', \eta; x, y) = {}_2F_1(\zeta, \varrho; \eta; x) \quad (1.8)$$

and

$$F_3(0, \zeta', \varrho, \varrho', \eta; x, y) = {}_2F_1(\zeta', \varrho'; \eta; y). \quad (1.9)$$

In this paper, we develop and study the image formulas involving the generalized Lommel–Wright function using fractional calculus integral operators. We use the generalized Marichev-Saigo-Maeda fractional integral operators, involving the Appell function, defined as follows:

$$\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa} f\right)(x) = \frac{x^{-\zeta}}{\Gamma(\kappa)} \int_0^x (x-t)^{\kappa-1} t^{-\zeta'} \times F_3\left(\zeta,\zeta',\varrho,\varrho';\kappa;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt, \quad \Re(\kappa)>0,\ \zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C},\ x>0 \quad (1.10)$$

and

$$\left(I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa} f\right)(x) = \frac{x^{-\zeta}}{\Gamma(\kappa)} \int_x^\infty (t-x)^{\kappa-1} t^{-\zeta} F_3\left(\zeta,\zeta',\varrho,\varrho';\kappa;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt, \quad \Re(\kappa)>0,\ \zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C},\ x>0. \quad (1.11)$$

respectively.

The power functions of left-hand sided and right-hand sided Marichev-Saigo-Maeda fractional integral operators as given in the Equations (1. 10) and (1. 11) (see Saigo et al. [6 , 20]) are given by

$$\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa} t^{\chi-1}\right)(x) = \frac{\Gamma(\chi)\Gamma(\chi+\kappa-\zeta-\zeta'-\varrho)\Gamma(\chi+\varrho'-\zeta')}{\Gamma(\chi+\varrho')\Gamma(\chi+\kappa-\zeta-\zeta')\Gamma(\chi+\kappa-\zeta'-\varrho)} x^{\chi+\kappa-\zeta-\zeta'-1}, \quad (1.12)$$

where ζ, ζ′, ϱ, ϱ′, κ ∈ ℂ, $x > 0$ and if ℜ(κ) > 0, ℜ(χ)> max{0, ℜ(ζ + ζ′ + ϱ − κ), ℜ(ζ′ − ϱ′)}.

(I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1) (x) = Γ (1 − χ − κ + ζ + ζ ′) Γ (1 − χ + ζ + ϱ ′ − κ) Γ (1 − χ − ϱ) Γ (1 − χ) Γ (1 − χ + ζ + ζ ′ + ϱ + ϱ ′ − κ) Γ (1 − χ + ζ − ϱ) × x χ − ζ − ζ ′ + κ − 1 , (1. 13)

where ζ, ζ′, ϱ, ϱ′, κ ∈ ℂ are such that ℜ(κ) > 0 and ℜ(χ) < 1 + min{ℜ(−ϱ), ℜ(ζ + ζ′ − κ), ℜ(ζ + ϱ′ − κ)}.

1. 1. Relation Among the Operators

In this section, we recall some relationships between the fractional integral operators.

If we set ζ′ = 0 then in view of the formula (1. 8), the relationship between Marichev-Saigo-Maeda and the Saigo fractional integral operators is found by Saxena and Saigo [6 , p. 93, Equation (2. 15)] as

(I 0 , x ζ , 0 , ϱ , ϱ ′ , η f) (x) = (I 0 , x η , ζ − η , − ϱ f) (x) , (ℜ (η) > 0) (1. 14)

and

(I x , ∞ ζ , 0 , ϱ , ϱ ′ , η f) (x) = (I x , ∞ η , ζ − η , − ϱ f) (x) , (ℜ (η) > 0) , (1. 15)

where the general operators I 0 , x ζ , 0 , ϱ , ϱ ′ , η and I 0 , x ζ , 0 , ϱ , ϱ ′ , η reduce, respectively, to the Saigo operators I 0 , x ζ , ϱ , η and I x , ∞ ζ , ϱ , η [17] defined as follows:

(I 0 , x ζ , ϱ , η f) (x) = x − ζ − ϱ Γ (ζ) ∫ 0 x (x − t) 2 ζ − 1 × F 1 (ζ + ϱ , − η ; ζ ; 1 − t x) f (t) d t , (ℜ (ζ) > 0) (1. 16) and

$$(I^{\zeta,\varrho,\eta}_{x,\infty} f)(x) = \int_x^\infty (t-x)^{\zeta-1} t^{-\zeta-\varrho} \, {}_2F_1\left(\zeta+\varrho, -\eta; \zeta; 1-\frac{x}{t}\right) f(t)\,dt, \quad (\Re(\zeta) > 0) \quad (1.17)$$

where integrals in (1. 16) and (1. 17) exist.

Let ζ, ϱ, η, χ ∈ ℂ with ℜ(ζ) > 0. Then the following power function formulas involving the Saigo operators hold true:

$$(I^{\zeta,\varrho,\eta}_{0,x} t^{\chi-1})(x) = \frac{\Gamma(\chi)\Gamma(\chi+\eta-\varrho)}{\Gamma(\chi-\varrho)\Gamma(\chi+\eta+\zeta)} x^{\chi-\varrho-1}, \quad \Re(\chi) > \max\{0, \Re(\varrho-\eta)\} \quad (1.18)$$

and

$$(I^{\zeta,\varrho,\eta}_{x,\infty} t^{\chi-1})(x) = \frac{\Gamma(1-\chi+\varrho)\Gamma(1-\chi+\eta)}{\Gamma(1-\chi)\Gamma(1-\chi+\zeta+\varrho+\eta)} x^{\chi-\varrho-1}, \quad (\Re(\chi) < 1 + \min\{\Re(\varrho), \Re(\eta)\}). \quad (1.19)$$

On replacing ϱ = −ζ in the operators $I^{\zeta,\varrho,\eta}_{0,x}(\cdot)$ and $I^{\zeta,\varrho,\eta}_{x,\infty}(\cdot)$, these reduce to the Riemann-Liouville and the Weyl fractional integral operators, respectively, by means of the following relationships (see Kilbas [11]):

$$(R^{\zeta}_{0,x} f)(x) = (I^{\zeta,-\zeta,\eta}_{0,x} f)(x) \quad (1.20)$$

and

$$(W^{\zeta}_{x,\infty} f)(x) = (I^{\zeta,-\zeta,\eta}_{x,\infty} f)(x). \quad (1.21)$$

The Riemann-Liouville fractional integral operator and the Weyl fractional integral operator are defined as follows (see e. g., [21]):

$$(R_{0,x}^{\zeta} f)(x) = \frac{1}{\Gamma(\zeta)} \int_0^x (x-t)^{\zeta-1} f(t)\, dt, \quad (\Re(\zeta) > 0) \quad (1.22)$$

and

$$(W_{x,\infty}^{\zeta} f)(x) = \frac{1}{\Gamma(\zeta)} \int_0^x (t-x)^{\zeta-1} f(t)\, dt, \quad (\Re(\zeta) > 0), \quad (1.23)$$

provided both the integrals converge.

The operators $I_{0,x}^{\zeta,\varrho,\eta}(\cdot)$ and $I_{x,\infty}^{\zeta,\varrho,\eta}(\cdot)$ reduce to Erdélyi–Kober fractional integral operators on setting $\varrho = 0$ as follows:

$$(E_{0,x}^{\zeta,\eta} f)(x) = (I_{0,x}^{\zeta,0,\eta} f)(x), \quad (1.24)$$

and

$$(K_{x,\infty}^{\zeta,\eta} f)(x) = (I_{0,x}^{\zeta,0,\eta} f)(x), \quad (1.25)$$

where the Erdélyi–Kober type fractional integral operators are defined as follows (see [22]):

$$(E_{0,x}^{\zeta,\eta} f)(x) = \frac{x^{-\zeta-\eta}}{\Gamma(\zeta)} \int_0^x (x-t)^{\zeta-1} t^{\eta} f(t)\, dt, \quad (\Re(\zeta) > 0) \quad (1.26)$$

and

$$(K_{x,\infty}^{\zeta,\eta} f)(x) = \frac{x^{\eta}}{\Gamma(\zeta)} \int_x^{\infty} (t-x)^{\zeta-1} t^{-\zeta-\eta} f(t)\, dt, \quad (\Re(\zeta) > 0), \quad (1.27)$$

The function $f(t)$ is constrained so that both the defining integrals (1. 26) and (1. 27) converge.

The *Beta transform* (see, e. g.[23]) of a complex valued function $f(t)$ of a real variable t is defined as follows:

$$B\{f(t): a, b\} = \int_0^1 t^{a-1} (1-t)^{b-1} f(t)\, dt, \quad \Re(t) > 0, \Re(a), \Re(b) > 0. \quad (1.28)$$

Beta transform of the power function $t^{\chi-1}$ is given by:

$$B\{t^{\chi-1}; a, b\} = \int_0^1 t^{a+\chi-2} (1-t)^{b-1} dt = \frac{\Gamma(a+\chi-1)\Gamma(b)}{\Gamma(a+\chi+b-1)}, \quad \Re(t) > 0, \Re(a), \Re(b) > 0. \quad (1.29)$$

The P_{δ} - *transform* of a complex valued function $f(t)$ of a real variable t denoted by $P_{\delta}[f(t); s]$ is a function $F(s)$ of a complex variable s, valid under certain conditions on $f(t)$, (given below is defined as (see Kumar [24])

$$P_{\delta}[f(t); s] = F(s) = \int_0^{\infty} [1 + (\delta-1)s]^{-\frac{t}{\delta-1}} f(t)\, dt, \quad \delta > 1. \quad (1.30)$$

Here $f(t)$ as a function is integrable over any finite interval (a, b), $0 < a < t < b$; there exists a real number c such that

(i) if $b > 0$ is arbitrary, then $\int_b^Y e^{-ct} f(t)\, dz$ tends to a finite limit as $Y \to \infty$

(ii) for arbitrary $a > 0$, ∫ ω a | f (t) d t | tends to a finite limit as ω → 0+, then the P_δ -transform $P_\delta[\, f(t); s\,]$ exists for ℜ (ln [1 + (δ - 1) s] δ - 1) > c for $s \in \mathbb{C}$.

P_δ - transform of the power function $t^{\chi - 1}$ is given by

P δ [z χ − 1 ; s] = { δ − 1 ln [1 + (δ − 1) s] } χ Γ (χ) , χ ∈ ℂ , ℜ (χ) > 0 , δ > 1. (1. 31)

P_δ -transform has found many applications. The pathway transforms are the paths going from the binomial form ln [1 + (δ - 1) s] - t δ - 1 to the exponential from e^{-st}. In P_δ -transform, the variable t is shifted from the binomial factor ln [1 + (δ - 1) s] - t δ - 1 to the exponent, Hence, this form is more suitable for obtaining translation, convolution, etc. Recently, Agarwal et al. [25] found the solution of non-homogeneous time fractional heat equation and fractional Volterra integral equation using integral transform of pathway type. Also, Srivastava et al. [26] and [27] found some results involving generalized hypergeometric function and generalized incomplete gamma function by using P_δ -transform.

If we take δ → 1 in Equation (1. 30), the P_δ -transform reduces to *Laplace integral transform* (Sneddon [23]):

L [f (t) ; s] = ∫ 0 ∞ e − t s f (t) d z ; , ℜ (s) > 0. (1. 32)

The following relationship between the P_δ -transform is defined by (1. 30) and the classical Laplace transform is defined by (1. 32)

$$P_\delta[f(t):s] = L\left[f(t):\frac{\ln[1+(\delta-1)s]}{\delta-1}\right], \quad (\delta>1) \quad (1.33)$$

or, equivalently,

$$L[f(t):s] = P_\delta\left[f(t):\frac{e^{(\delta-1)s}-1}{\delta-1}\right], \quad (\delta>1), \quad (1.34)$$

which can be applied to convert the table of Laplace transforms into the corresponding table of P_δ-transforms and *vice versa* .

The following integral formula involving the Whittaker function (see Mathai et al. [16 , p. 56]) is used in finding the image formula:

$$\int_0^\infty t^{\tau-1} e^{-\frac{1}{2}} W_{\sigma,\eta}(t)\,dt = \frac{\Gamma(\tau+\eta+\frac{1}{2})\,\Gamma(\tau-\eta+\frac{1}{2})}{\Gamma(\tau-\sigma+\frac{1}{2})}, \quad (\sigma\in\mathbb{C}, \Re(\tau\pm\eta)>-1/2). \quad (1.35)$$

The Whittaker function (see e. g., Mathai et al. [16 , p. 22]) is defined by

$$W_{\sigma,\eta}(z) = \frac{\Gamma(-2\eta)}{\Gamma(\frac{1}{2}-\sigma-\eta)} M_{\sigma,\eta}(z) + \frac{\Gamma(2\eta)}{\Gamma(\frac{1}{2}-\sigma+\eta)} M_{\sigma,-\eta}(z) = W_{\sigma,-\eta}(z), \quad \sigma\in\mathbb{C}, \Re(1/2+\eta\pm\sigma)>0 \quad (1.36)$$

where

$$M_{\sigma,\eta}(z) = z^{\eta+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(\frac{1}{2}-\sigma+\eta; 2\eta+1; z\right), \quad \Re(1/2+\eta\pm\sigma)>0, \ |\arg z|<\pi. \quad (1.37)$$

2. Image Formula Associated With Fractional Integral Operators

Here, we establish image formulas for the generalized Lommel–Wright function involving Saigo-Maeda fractional integral operators (1. 10) and (1. 11), in terms of the Fox–Wright function.

Theorem 2. 1. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ ∈ ℂ, *m* ∈ ℕ, φ > 0 *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (χ + ω) > max { 0 , ℜ (ζ + ζ ′ + ϱ − κ) , ℜ (ζ ′ − ϱ ′) } (2. 1)

then there holds the formula

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (t z)] (x) = x A − ζ − ζ ′ + κ − 1 (z 2) ω + 2 ϑ 4 ψ 4 + m [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) ⏟ m − t i m e s | − (z x) 2 4] (2. 2)

where A = χ + ω + 2ϑ.

Proof: Under the conditions stated with the Theorem 2. 1, by taking the fractional integral of (1. 1) using the equation (1. 10) therein and changing the order of integration and summation, we get

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (t z)] (x) = ∑ k = 0 ∞ (− 1) k (z 2) ω + 2 ϑ + 2 k Γ (k + 1) (Γ (ϑ + k + 1)) m Γ (ω + k φ + ϑ + 1) k ! × I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t ω + 2 ϑ + 2 k + χ − 1) (x) (2. 3)

Further, applying the result (1. 12) with χ replaced by χ + ω + 2ϑ + 2 *k* , we obtain

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (t z)] (x) = x A − ζ − ζ ′ + κ − 1 (z 2) ω + 2 ϑ ∑ k = 0 ∞ (− 1) k Γ (A + 2 k) Γ (k + 1) Γ (A + ϱ ′ + 2 k) (Γ (ϑ + 1 + k)) m × Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (ω + ϑ + 1 + φ k) Γ (A + κ − ζ − ζ ′ + 2 k) × (z x) 2 k 4 k k ! (2. 4)

Here $A = \chi + \omega + 2\vartheta$.

Interpreting the right-hand side of the above equation, in view of the definition (1. 2), we arrive at the result (2. 2).

Theorem 2. 2. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ ∈ ℂ, *m* ∈ ℕ, φ > 0 *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (χ − ω) > 1 + min { ℜ (− ϱ) , ℜ (ζ + ζ ′ − κ) , ℜ (ζ + ϱ ′ − κ) } (2. 5)

then there holds the formula

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (z / t)] (x) = x κ − ζ − ζ ′ − A (z 2) ω + 2 ϑ 4 ψ 4 + m [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) ⏟ m − t i m e s | − z 2 4 x 2] (2. 6)

where A = 1 − χ + ω + 2ϑ.

Proof: Under the conditions stated with the Theorem 2. 2, on making use of the definitions (1. 11) and (1. 1) and changing the order of integration and summation, we have

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (z / t)] (x) = ∑ k = 0 ∞ (− 1) k (z 2) ω + 2 ϑ + 2 k Γ (k + 1) (Γ (ϑ + k + 1)) m Γ (ω + k φ + ϑ + 1) k ! × I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − ω − 2 ϑ − 2 k − 1) (x) (2. 7)

Here, on applying the formula (1. 13) with χ replaced by χ − ω − 2ϑ − 2 *k* , we obtain

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , m (z / t)] (x) = x κ − ζ − ζ ′ − A (z 2) ω + 2 ϑ ∑ k = 0 ∞ (− 1) k Γ (A − ϱ + 2 k) Γ (A + 2 k) (Γ (ϑ + k + 1)) m × Γ (k + 1) Γ (A − κ + ζ + ζ ′ + 2 k) Γ (A + ζ + ϱ ′ − κ + 2 k) Γ (A + ζ − ϱ + 2 k) Γ (ω + k φ + ϑ + 1) Γ (A + ζ + ζ ′ + ϱ ′ − κ + 2 k) × (z) 2 k (4 x 2) k k ! (2. 8)

where $A = 1 - \chi + \omega + 2\vartheta$.

So in view of the definition of the generalized Lommel–Wright function given by (1. 1), the Equation (2. 8) leads to the result (2. 6).

For $m = 1$ and in the light of Equation (1. 3), Theorem 2. 1 leads to the following corollaries:

Corollary 2. 1. *Under the conditions stated with the Equation (2. 1), the following image formula*

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , 1 (z t)] (x) = x A − ζ − ζ ′ + κ − 1 (z 2) ω + 2 ϑ × 4 ψ 5 [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) | − (z x) 2 4] (2. 9)

$A = \chi + \omega + 2\vartheta$, *for generalized Bessel function* J ω , ϑ φ , 1 (z t) *holds true* .

Corollary 2. 2. *Under the conditions stated with the Equation (2. 5), the image formula*

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω , ϑ φ , 1 (z / t)] (x) = x κ − ζ − ζ ′ − A (z 2) ω + 2 ϑ × 4 ψ 5 [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) | − z 2 4 x 2] (2. 10)

A = 1 − χ + ω + 2ϑ, *for generalized Bessel function* J ω , ϑ φ , 1 (z / t) *holds true* .

If we take *m* = 1, φ = 1, and ϑ = 1 2 in (2. 2), then we obtain the corresponding results for the Struve function $H_\omega(\cdot)$ [16] as

Corollary 2. 3. *Under the conditions stated with the Equation (2. 1), the following image formula*

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 H ω (z t)] (x) = x A − ζ − ζ ′ + κ − 1 (z 2) ω + 1 × 4 ψ 5 [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + 3 2 , 1) , (3 2 , 1) | − (z x) 2 4] (2. 11)

A = χ + ω + 1, *for Struve function* H_ω (*zt*) *holds true* .

Corollary 2. 4. *Under the conditions stated with the Equation (2. 5), the following image formula*

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 H ω (z / t)] (x) = x χ − ω − ζ − ζ ′ + κ − 2 (z 2) ω + 1 × 4 ψ 5 [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + 3 2 , 1) , (3 2 , 1) | − z 2 4 x 2] (2. 12)

where A = 2 − χ + ω, *for Struve function* H_ω (*z* / *t*) *holds true* .

2. 1. Special Cases

(1) On taking φ = 1, *m* = 1, ϑ = 0, and *z* = 1 in Theorem 2. 1, we obtain the image formula for the Bessel function considered by Purohit et al. [28 , Theorem 1].

Corollary 2. 5. *Under the conditions stated with the Equation (2. 1), the following image formula*

I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω (t)] (x) = x χ + ω − ζ − ζ ′ + κ − 1 2 ω × 3 ψ 4 [(χ + ω , 2) , (χ + ω + κ − ζ − ζ ′ − ϱ , 2) , (χ + ω + ϱ ′ − ζ ′ , 2) (χ + ω + ϱ ′ , 2) , (χ + ω + κ − ζ − ζ ′ , 2) , (χ + ω + κ − ζ ′ − ϱ , 2) , (ω + 1 , 1) | − x 2 4] (2. 13)

for Bessel function J_ω (*t*) *holds true* .

(2) Further, on taking φ = 1, *m* = 1, and ϑ = 0 in Theorem 2. 2, we arrive the right-sided image formula for the Bessel function considered by Purohit et al. [28 , Theorem 2].

Corollary 2. 6. *Under the conditions stated with the Equation (2. 5), the image formula*

I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ [t χ − 1 J ω (1 / t)] (x) = x κ − ζ − ζ ′ − 1 + χ − ω 2 ω × 3 ψ 4 [(1 − χ + ω − κ + ζ + ζ ′ , 2) , (1 − χ + ω + ζ + ϱ ′ − κ , 2) , (1 − χ + ω − ϱ , 2) (1 − χ + ω , 2) (1 − χ + ω + ζ + ζ ′ + ϱ ′ − κ , 2) , (1 − χ + ω + ζ − ϱ , 2) , (ω + 1 , 1) | − 1 4 x 2] (2. 14)

for Bessel function J_ω (1/ *t*) *holds true* .

3. Image Formulas Associated With Integral Transforms

In this section, we obtain the theorem involving the results obtained in previous sections associated with the integral transforms such as Beta transform, pathway transform, Laplace transform, and Whittaker transform.

3. 1. Image Formulas for Beta Transform

Theorem 3. 1. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ ∈ ℂ, *m* ∈ ℕ, φ > 0, *and x* > 0 *be such that*

ℜ (l) > 0 , ℜ (n) > 0 ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (χ + ω) > max { 0 , ℜ (ζ + ζ ′ + ϱ − κ) , ℜ (ζ ′ − ϱ ′) } (3. 1)

then the following Beta transform formula holds :

B [I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (t z)) (x) : l , n] = x A − ζ − ζ ′ + κ − 1 Γ (n) 2 ω + 2 ϑ 5 ψ 5 + m [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (C − n , 2) (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (C , 2) , (ϑ + 1 , 1) □ m − t i m e s | − x 2 4] (3. 2)

Here A = χ + ω + 2ϑ *and C* = *l* + ω + 2ϑ + *n* .

Proof: For our convenience, let the left-hand side of the formula (3. 2) be denoted by ς. Applying (1. 28) to Equation (3. 2), we get

ς = ∫ 0 1 z l − 1 (1 − z) n − 1 [I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (t z)) (x)] d z .

Here, applying Equation (2. 2) to the integral, we obtain the following expression ς = ∫ 0 1 z l − 1 (1 − z) n − 1 z ω + 2 ϑ x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ × ∑ k = 0 ∞ (− 1) k Γ (A + 2 k) Γ (k + 1) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) × Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m × (z x 2) k 4 k k ! d z

Here A = χ + ω + 2ϑ.

Interchanging the order of integration and summation, we have

ς = x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ Γ (A + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) Γ (A + κ − ζ ′ − ϱ + 2 k) × Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) (− 1) k Γ (A + ϱ ′ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m (x 2) k 4 k k ! × ∫ 0 1 z l + ω + 2 ϑ + 2 k − 1 (1 − z) n − 1 d z = x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ Γ (l + ω + 2 ϑ + 2 k) Γ (n) Γ (A + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (l + ω + 2 ϑ + 2 k + n) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) × Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m × (− x 2) k 4 k k ! (3. 3)

Interpreting the right-hand side of the above equation, in the view of the definition (1. 2), we arrive at the required result (3. 2).

Theorem 3. 2. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ, ω ∈ ℂ, *m* ∈ ℕ, φ > 0, *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (l) > 0 , ℜ (n) > 0 , ℜ (χ − ω) > 1 + min { ℜ (− ϱ) , ℜ (ζ + ζ ′ − κ) , ℜ (ζ + ϱ ′ − κ) } (3. 4)

then the following Beta transform formula holds :

B [I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (z / t)) (x) : l , n } = x κ − ζ − ζ ′ − A Γ (n) 2 ω + 2 ϑ × 5 ψ 5 + m [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (C − n , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (C , 2) , (ϑ + 1 , 1) ⏟ m − t i m e s | − 1 4 x 2] (3. 5)

where A = 1 − χ + ω + 2ϑ and *C* = *l* + ω + 2ϑ + *n* .

Proof: The proof of the fractional integral formula (3. 5) is similar to the proof of the formula (3. 2) given in Theorem 3. 1.

Remark 3. 1.

(1) *For m* = 1, *Theorem 3. 1 and Theorem 3. 2 leads to the corresponding results for fractional integral of generalized Bessel function defined by (1. 3)* .

(2) *If we take m* = 1, φ = 1, *and* ϑ = 1 2 *in (3. 2) and (3. 5), we get the corresponding results for fractional integral of Struve function defined in (1. 4)* .

(3) *On taking m* = 1, φ = 1, *and* ϑ = 0, *in (3. 2) and (3. 5), we get the results for fractional integral of Bessel function defined in (1. 5)* .

3. 2. Image Formulas for *P* δ -Transform

Theorem 3. 3. *Let* ζ, ζ′, ϱ, ϱ′, κ, χ, ϑ ∈ ℂ, *m* ∈ ℕ, φ > 0, ℜ(χ) > 0, ℜ(*s*) > 0, δ > 1, *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (s) > 0 , ℜ (χ + ω) > max { 0 , ℜ (ζ + ζ ′ + ϱ − κ) , ℜ (ζ ′ − ϱ ′) } (3. 6) *then the following P* δ - *transform formula holds* :

P δ [z l − 1 (I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1 J ω , ϑ φ , m (t z)) (x) : s] = (Λ (δ ; s)) l + ω + 2 ϑ x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ × 5 ψ 4 + m [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (l + ω + 2 ϑ , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) □ m − t i m e s | − (Λ (δ ; s) x) 2 4] (3. 7)

where A = χ + ω + 2ϑ *and* Λ (δ ; s) = (δ - 1 ln [1 + (δ - 1) s]) .

Proof: For our convenience, we let the left-hand side of the formula (3. 7) be denoted as Ξ. Applying (1. 30) to Equation (3. 2) we get,

Ξ = ∫ 0 ∞ [1 + (δ − 1) s] − z δ − 1 z l − 1 I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (t z)) (x)] d z

Here, applying Equation (2. 4) to the integral, we obtain the following expression:

Ξ = x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ (− 1) k Γ (A + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m (x) 2 k 4 k k ! × ∫ 0 ∞ [1 + (δ − 1) s] − z δ − 1 z ω + 2 ϑ + 2 k + l − 1 d z

Here making use of the result (1. 31) and interchanging the order of integration and summation, we obtain, Ξ = (Λ (δ ; s)) l + ω + 2 ϑ x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ Γ (A + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) × Γ (ω + 2 ϑ + 2 k + l) Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) (− 1) k Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m { Λ (δ ; s) x } 2 k 4 k k ! (3. 8)

where *A* = χ + ω + 2ϑ and Λ (δ ; s) = (δ - 1 ln [1 + (δ - 1) s]) .

In view of the definition (1. 2), we arrive at the required result (3. 7).

Theorem 3. 4. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ ∈ ℂ, *m* ∈ ℕ, φ > 0ℜ(χ) > 0, ℜ(*s*) > 0, δ > 1, *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (s) > 0 , ℜ (χ − ω) > 1 + min { ℜ (− ϱ) , ℜ (ζ + ζ ′ − κ) , ℜ (ζ + ϱ ′ − κ) } (3. 9)

then the following P δ *- transform formula holds* :

P δ (z l − 1 [I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1 J ω , ϑ φ , m (z / t)] (x) : s) = (Λ (δ ; s)) l + ω + 2 ϑ x χ − ω − 2 ϑ − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ × 5 ψ 4 + m [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (l + ω + 2 ϑ , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) m − t i m e s | − { Λ (δ ; s) } 2 4 x 2] (3. 10)

where A = 1 − χ + ω + 2ϑ *and* Λ (δ ; s) = { δ - 1 ln [1 + (δ - 1) s] } .

Proof: Our demonstration of the *P* δ -transform of generalized Lommel–Wright function (3. 10) is based upon the known result (2. 6).

A limit case of the Theorems 3. 3 and 3. 4 when δ → 1 yields the following corollaries for the Laplace transform in view of the (1. 32).

Corollary 3. 1. *Under the conditions stated with the Equation (3. 6), the following Laplace transform formula holds true* :

P δ (z l − 1 (I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1 J ω , ϑ φ , m (t z)) (x) : s) = x A − ζ − ζ ′ + κ − 1 s l 2 ω + 2 ϑ × 5 ψ 4 + m [(A , 2) , (A + κ − ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (l + ω + 2 ϑ , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) ︸ m − t i m e s | − x 2 s 2 l 4] (3. 11)

where A = χ + ω + 2ϑ.

Corollary 3. 2. *Under the conditions stated with the Equation (3. 9), the following Laplace transform formula holds true* :

P δ (z l − 1 [I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1 J ω , ϑ φ , m (z / t)] (x) : s) = x χ − ω − 2 ϑ − ζ − ζ ′ + κ − 1 s l 2 ω + 2 ϑ × 5 ψ 4 + m [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (l + ω + 2 ϑ , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (ϑ + 1 , 1) ︸ m − t i m e s | − 1 s 2 l 4 x 2] (3. 12)

where A = 1 − χ + ω + 2ϑ.

Remark 3. 2.

(1) *On taking m* = 1, *Theorems 3. 3 and 3. 4 lead to the P* $_{δ}$ - *transform formulas for fractional integrals of generalized Bessel function* .

(2) *A limit case of the Theorems 3. 3 and 3. 4, when* δ → 1 *and m* = 1, *yields the Laplace transform formulas for fractional integrals of generalized Bessel function* .

(3) *On taking m* = 1, φ = 1, *and* ϑ = 1 2 , *Theorems 3. 3 and 3. 4 yield the P* δ - *transform formulas for fractional integrals of Struve function* .

(4) *A limit case of Theorem 3. 3 and 3. 4, when* δ → 1 *and m* = 1, φ = 1, *and* ϑ = 1 2 , *yield the Laplace transform formulas for fractional integrals of Struve function* .

(5) *On taking m* = 1, φ = 1, *and* ϑ = 0, *Theorem 3. 3 and 3. 4 yield the corresponding results for fractional integrals of Bessel function* .

(6) *A limit case of Theorem 3. 3 when* δ → 1 *and m* = 1, φ = 1, *and* ϑ = 0 *yield the corresponding Laplace transform formulas for fractional integrals of Bessel function* .

3. 3. Image Formulas for Whittaker Transform

Theorem 3. 5. *Let* ζ, ζ′, ϱ, ϱ′, κ, ϑ, η, σ ∈ ℂ, *m* ∈ ℕ, φ > 0, *and x* > 0 *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (τ ± η) > − 1 / 2 , ℜ (χ + ω) > max { 0 , ℜ (ζ + ζ ′ + ϱ − κ) , ℜ (ζ ′ − ϱ ′) (3. 13)

then the following Whittaker transform formula holds :

∫ 0 ∞ z σ − 1 e − z / 2 [W σ , η I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (z t)) (x)] d z = x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ 6 ψ 5 + m [(A , 2) , (A + κ

− ζ − ζ ′ − ϱ , 2) , (A + ϱ ′ − ζ ′ , 2) , (E + η , 2) , (E − η , 2) , (1 , 1) (A + ϱ ′ , 2) , (A + κ − ζ − ζ ′ , 2) , (A + κ − ζ ′ − ϱ , 2) , (ω + ϑ + 1 , φ) , (E − σ , 2) , (ϑ + 1 , 1) ⏟ m − t i m e s | − x 2 4] (3. 14)

where A = χ + ω + 2ϑ *and E* = τ + ω + 2ϑ + 1/2.

Proof: For simplicity, let ϖ be the left-hand side of the formula (3. 14). Applying (1. 35) to Equation (3. 14), we have

ϖ = ∫ 0 ∞ z σ − 1 e − z / 2 W σ , η [I 0 + ζ , ζ ′ , ϱ , ϱ ′ , κ (t χ − 1 J ω , ϑ φ , m (z t)) (x)] d z . (3. 15)

Here, applying Equation (2. 2) to the integral, we obtain the following expression:

ϖ = ∫ 0 ∞ z σ + ω + 2 ϑ − 1 e − z / 2 W σ , η [x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ Γ (A + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) × × Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) (− 1) k Γ (A + κ − ζ ′ − ϱ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m × (z x) 2 k 4 k k !] d z

where *A* = χ + ω + 2ϑ. Interchanging the order of integration and summation, we have

ϖ = x A − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ ∑ k = 0 ∞ Γ (E + η + 2 k) Γ (E − η + 2 k) Γ (A + κ − ζ − ζ ′ − ϱ + 2 k) Γ (E − σ + 2 k) Γ (A + κ − ζ ′ − ϱ + 2 k) × (− 1) k Γ (A + 2 k) Γ (A + ϱ ′ − ζ ′ + 2 k) Γ (k + 1) Γ (A + ϱ ′ + 2 k) Γ (A + κ − ζ − ζ ′ + 2 k) Γ (ω + ϑ + 1 + φ k) (Γ (ϑ + 1 + k)) m x 2 k 4 k k ! (3. 16) where $A = \chi + \omega + 2\vartheta$ and $E = \tau + \omega + 2\vartheta + 1/2$.

Interpreting the right-hand side of the above equation, in view of the definition (1. 2), we arrive at the required result (3. 14).

Theorem 3. 6. *Let* $\zeta, \zeta', \varrho, \varrho', \kappa, \vartheta, \eta, \sigma \in \mathbb{C}$, $m \in \mathbb{N}$, $\varphi > 0$, *and* $x > 0$ *be such that*

ℜ (κ) > 0 , ℜ (ω) > − 1 , ℜ (τ ± n) > − 1 / 2 , ℜ (χ − ω) > 1 + min { ℜ (− ϱ) , ℜ (ζ + ζ ′ − κ) , ℜ (ζ + ϱ ′ − κ) } (3. 17)

then there holds the formula

∫ 0 ∞ z σ − 1 e − z / 2 W σ , η [(I 0 − ζ , ζ ′ , ϱ , ϱ ′ , κ t χ − 1 J ω , ϑ φ , m (z t)) (x)] d z = x χ − ω − 2 ϑ − ζ − ζ ′ + κ − 1 2 ω + 2 ϑ 6 ψ 5 + m [(A − κ + ζ + ζ ′ , 2) , (A + ζ + ϱ ′ − κ , 2) , (A − ϱ , 2) , (E + η , 2) , (E − η , 2) , (1 , 1) (A , 2) (A + ζ + ζ ′ + ϱ ′ − κ , 2) , (A + ζ − ϱ , 2) , (ω + ϑ + 1 , φ) , (E − σ , 2) , (ϑ + 1) ︸ m − t i m e s | − 1 4 x 2] (3. 18)

where $A = 1 - \chi + \omega + 2\vartheta$ *and* $E = \tau + \omega + 2\vartheta + 1/2$.

Proof: We can establish the result given in Theorem 3. 6 similar to the proof of Theorem 3. 5.

Remark 3. 3.

(1) *For* $m = 1$, *Theorems 3. 5 and 3. 6 lead to the corresponding results for fractional integral of generalized Bessel function defined in (1. 3)* .

(2) *If we take m* = 1, φ = 1, *and* ϑ = 1 2 , *Theorems 3. 5 and 3. 6 yield the corresponding results for fractional integral of Struve function defined in (1. 4)* .

(3) *On taking m* = 1, φ = 1, *and* ϑ = 0, *Theorems 3. 5 and 3. 6 yield the corresponding results for fractional integral of Bessel function defined in (1. 5)* .

4. Special Cases and Concluding Remarks

In this section, we consider some special cases of our main results involved in Theorems 2. 1–3. 6 which can be obtained by setting ζ′ = 0. These interesting corollaries of our results involve the Saigo fractional integral operators I 0 , x ζ , ϱ , η and I x , ∞ ζ , ϱ , η and can be deduced from the Theorems 2. 1–3. 6 by appropriately applying the relationships given in the definitions (1. 16) and (1. 17). If we set ϱ = −ζ in the Theorems 2. 1–3. 6, then from the relationships (1. 20) and (1. 21) we obtain the corresponding results for the Riemann–Liouville and the Weyl fractional integral operators, respectively. Again, if we put ϱ = 0 in the Theorems 2. 1–3. 6, then from the relationships (1. 24) and (1. 25) we obtain the analogous results for Erdélyi-Kober type fractional integral operators.

In our present investigation, we establish the relationship between well-known fractional integral operators with novel integral transforms. The results obtained here are useful in deriving at various image formulas. The results presented here are very generic and can be specialized to give further potentially interesting and useful formulas involving fractional integral operators.

Author Contributions

RPA devised the problem and supervised the manuscript by adding various results to it. RA and SJ worked on the mathematics in the manuscript. DB provided guidance, checked all calculations, and suggested language modifications to the article paper.

Conflict of Interest Statement

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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