Arithmetic vs. geometric means: empirical

1 " Arithmetic vs. Geometric Means: Empirical Evidence and Theoretical Issues" by Jay B. Abrams, ASA, CPA, MBA Copyright 1996 There has been a flurry of articles about the relative merits of using the arithmetic mean (AM) versus the geometric mean (GM). The Ibbotson SBBI Yearbook took the first position that the arithmetic mean is the correct mean to use in valuation. Allyn Joyce's June 1995 BVR article initiated arguments for the GM as the correct mean.

The previous articles have centered around Professor Ibbotson's famous example using a binomial distribution with 50%-50% probabilities of a +30% and -10% return. The debate has been very interesting, but it is off on a tangent, focused on the wrong issue. There are theoretical and empirical reasons why the arithmetic mean is the correct one. We will look at both in this article. Theoretical Superiority of Arithmetic Mean Rather than argue about Ibbotson's much debated above example, I prefer to cite and elucidate another quote from his book: In general, the geometric mean for any time period is less than or equal to the arithmetic mean.

The two means are equal only for a return series that is constant (i. . , the same return in every period). For a non-constant series, the difference between the two is positively related to the variability or standard deviation of the returns.

For example, in Table 6-7, the difference between the arithmetic and geometric mean is much larger for risky large company stocks than it is for nearly riskless Treasury bills. 1 The GM measures the magnitude of the returns, as the investor starts with one portfolio value and ends with another.

It does not measure the variability of the journey, as does the AM. As noted in SBBI, the GM is backward looking. There is no difference in the GM of two stocks (or portfolios), one of which is highly volatile and the other of which is absolutely 1 SBBI-1996, p. 104 2 stable, while the AM is forward looking in that it does impound the volatility of the stocks.

As Mark Twain said, “ Forecasting is difficult—especially into the future. ” Had Twain been acquainted with portfolio theory, he obviously would have been an AM advocate. I suppose the GM advocates could cite Paul McCartney on their side for his song Yesterday. I would then have to counter with Steven Spielberg and Back To The Future. Table I contains an illustration of two stock series. The first one is highly volatile, with a standard deviation of returns of 65%, while the second one has a zero standard deviation.

It makes no sense intuitively that the GM is the correct one. That would imply that both stocks are equally risky, since they have the same GM. Would anyone really consider stock #2 equally as risky as #1? If so, let’s trade stocks! Every modern model to calculate discount rates recognizes that investors are risk averse and avoid volatility unless they are adequately compensated for undertaking it. It is more consistent to use the mean that fully impounds risk (AM) than the one that has had risk removed from it (GM). Another aspect of consistency in favor of the AM can be found in my article on the size effect, 2 which demonstrates that stock market portfolio returns correlate very well (98% R2) with the standard deviation of returns, i. e.

, risk. Which mean returns correlate best with the volatility of returns? Obviously the AM. In other words, the dependent variable (AM returns) is consistent with the independent variable (standard deviation of returns) in the regression. The latter is risk, and the former is the fully risk-impounded rate of return. GM is less consistent.

Using CAPM leads us to the same conclusions vis-a-vis AM vs. GM. The equation still says return is some function of risk. It is more consistent to use a fully-riskimpounded return than a risk-neutered return. In the next section, we will confirm this empirically using my Log Size Model.

Table II: Empirical EvidenceTable II contains both the geometric and arithmetic means for the Ibbotson deciles for 1926-1995 data and regressions of those returns against the standard deviation of returns and the natural logarithm of the average market capitalization (don't ask whether it is an arithmetic or geometric average size!) of the firms in the decile. Regression #1: Returns as a Function of Risk The arithmetic mean outperforms3 the geometric mean in this regression, with Adjusted R2 of 97. 78% versus 81. 93% and t-statistic of 19. 9 versus 6.

5. Additionally, the constant of 2 " A Breakthrough in Calculating Reliable Discount Rates," Valuation, August, 1994. Additionally, Grabowski and King subsequently published articles on the same topic in Business Valuation Review, June 1995, p. 69-74 and September 1996, p.

103-115. 3 In other words, AM does a better job of explaining risk than GM. 3 5. 24% for the arithmetic mean makes economic sense, i. e. , it matches the 70-year average Treasury Bond rate—the best proxy for the risk-free rate—

while the constant for the geometric mean matches nothing in particular; and how could it? The geometric mean is stripped of some of the information relating to volatility, so therefore it will obviously do an inferior job of forecasting returns based on volatility.

Let’s see how it does in explaining returns as a function of log size. Regression #2: Returns as a Function of Log Size Again, the arithmetic mean outperforms the geometric mean. Its adjusted R2 is 91. 43% compared to 84. 48% for the geometric mean.

The absolute value of Its t-statistic is 9. 2, compared to 6. 6 for the geometric mean. Only in one measure, the standard error of the estimate, is the geometric mean superior. However, the adjusted R2 incorporates the standard error of the Y estimate and is a more significant measure. It is not that the GM is a terrible measure of risk.

Its R-Squares in the various regressions are quite decent. It is just not as good as the AM. It clearly underperforms the AM, the reason being that useful information about risk has been stripped away. Regression #2A: Same as #2 Using Deciles #1-9 Only Regression #2A repeats the results from Regression #2, except that we treat Decile #10 as an outlier.

The arithmetic and geometric mean R Squares are 98. 67% and 79. 08%, respectively, a very large difference in favor of AM. The t-statistics are 24. 3 and 5.

6, respectively, again with a significant advantage to the arithmetic mean. ConclusionIt is clear that there are substantial theoretical and empirical advantages that point to the arithmetic mean as the correct measure of return in the market for calculating discount rates. 4 George Cassiere does bring an intriguing argument that the buy-and-hold strategy obviates the necessity to deal with volatility. 4 It is true that buyers of privately-held businesses are usually in it for the long term, which is not necessarily true of investors in the stock market. I agree that investors in privately-held businesses do not live and die with each year’s returns and volatility. Nevertheless, most businesses that survive sell every 10 years or so.

An investor would be most foolish to ignore the volatility of the business. Any rational investor should be concerned that a medical emergency or other life circumstances could at a moment’s notice force him to sell quickly. The more volatile the business, the greater the probability of a disaster, and most buyers of privately-held businesses are risk-averse like the rest of humanity. On a more personal level, buyers will prefer a firm that makes a steady $50, 000 per year for the owner rather than one that alternates between $0 and $100, 000.

A risk-averse buyer will pay more for the first firm than the second one. We must remember that volatile returns probably indicate volatile income—earnings surprises are a major cause of price volatility—which buyers will avoid unless properly compensated. It is very hard to believe that buyers will ignore volatility, especially over the long run. Now that we see the advantages of the AM, it is useful to note that the greater divergence between the AM and GM as firm size decreases and volatility increases means that using the GM results in overvaluation that is inversely related to size, i. e. using the GM on a small firm will cause a greater percentage overvaluation than using the GM on a large firm.

4 “ Geometric Mean Return Premium Versus the Arithmetic Mean Return Premium-Expanding on the SBBI 1995 Yearbook Examples, Business Valuation Review, March, 1996. Table I Geometric Vs. Arithmetic Returns (1) (2) (3) (4) (5) Stock (or Portfolio) #1 Stock (or Portfolio) #2 Year Price Annual Return Price Annual Return 0 $100. 00 NA $100. 00 NA 1 $150.

00 50. 0000% $111. 61 11. 6123% 2 $68. 00 -54.

6667% $124. 57 11. 6123% 3 $135. 00 98.

5294% $139. 04 11. 6123% 4 $192. 00 42.

2222% $155. 18 11. 6123% $130. 00 -32.

2917% $173. 21 11. 6123% 6 $79. 00 -39. 2308% $193.

32 11. 6123% 7 $200. 00 153. 1646% $215.

77 11. 6123% 8 $180. 00 -10. 0000% $240.

82 11. 6123% 9 $250. 00 38. 8889% $268.

79 11. 6123% 10 $300. 00 20. 0000% $300. 00 11. 6123% Standard Deviation 64.

9139% 0. 0000% Arithmetic Mean 26. 6616% 11. 6123% Geometric Mean 11. 6123% 11.

6123% Table II Geometric Vs. Arithmetic Returns NYSE Data By Decile & Statistical Analysis: 1926-1995 (1) (2) (3) (4) (5) (6) (7) (8) (9) Geometric Arithmetic Recent Mkt Decile Mean Mean Return Std Dev Capitalization % Cap # Co. s Avg Cap= FMV Ln(FMV) 1 9. 69% 11.

2% 18. 95% 3, 110, 306, 745, 000 64. 23% 169 18, 404, 181, 923 23. 6358 2 10.

94% 13. 36% 22. 55% 743, 402, 451, 000 15. 35% 169 4, 398, 831, 071 22.

2046 3 11. 39% 14. 07% 24. 39% 384, 020, 909, 000 7. 93% 170 2, 258, 946, 524 21.

5382 4 11. 46% 14. 65% 26. 84% 226, 702, 002, 000 4. 68% 169 1, 341, 431, 964 21. 0170 5 12.

09% 15. 60% 27. 67% 146, 129, 715, 000 3. 02% 169 864, 672, 870 20.

5779 6 11. 71% 15. 53% 28. 72% 98, 979, 665, 000 2. 04% 169 585, 678, 491 20. 1883 7 11.

68% 15. 98% 31. 18% 64, 087, 771, 000 1. 32% 170 376, 986, 888 19.

7477 8 11. 94% 17. 11% 35. 01% 39, 063, 761, 000 0.

81% 169 231, 146, 515 19. 2586 9 12. 08% 17. 86% 37. 6% 21, 589, 252, 000 0.

45% 169 127, 747, 053 18. 6656 10 13. 83% 22. 04% 46. 81% 8, 220, 123, 000 0. 17% 170 48, 353, 665 17.

6941 Std Dev 1. 04% 2. 88% 1, 693 Value Wtd Index 11. 86% 20.

24% 4, 842, 502, 394, 000 100. 00% Regression #1: Return = f(Std Dev. of Returns) Regression #2: Return = f [Ln(FMV)] Geometric Geometric Arithmetic Mean Mean Arithmetic Mean Mean Constant 5. 24% 8.

17% Constant 47. 94% 22. 82% Std Err of Y Est 0. 43% 0.

44% Std Err of Y Est 0. 89% 0. 43% R Squared 98. 02% 83. 93% R Squared 91.

43% 84. 48% Adjusted R Squared 97. 78% 81. 93% Adjusted R Squared 90.

36% 82. 54% No. of Observations 10 10 No. f Observations 10 10 Degrees of Freedom 8 8 Degrees of Freedom 8 8 X Coefficient(s) 35.

11% 11. 70% X Coefficient(s) -1. 57% -0. 54% Std Err of Coef. 1. 76% 1.

81% Std Err of Coef. 0. 17% 0. 08% T 19. 9 6. 5 T -9.

2 -6. 6 P < . 01% < . 01% P < . 01% < .

01% Regression #2A: Return = f [Ln(FMV)] First 9 Deciles Only Geometric Arithmetic Mean Mean Constant 41. 26% 20. 59% Std Err of Y Est 0. 23% 0. 34% R Squared 98. 83% 81.

70% Adjusted R Squared 98. 67% 79. 08% No. of Observations 9 9 Degrees of Freedom 7 7 X Coefficient(s) -1. 26% -0.

44% Std Err of Coef. 0. 05% 0. 08% T -24. 3 -5. 6 P < .

01% < . 01%