

A new approach to  
the goldbach  
conjecture philosophy  
essay



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The Goldbach conjecture has been approached using methods as complex as the circle method and the sieve method. After extensive research, I found out that there were several approaches to this conjecture, each approach building on a previous one or providing an alternate point of view. However, no one had actually attempted to look at the conjecture from an algebraic point of view.

By setting up initial conjectures such as the general expression of odd and even numbers, and proving these conjectures, I set up a base for my proof for the Goldbach conjecture. Afterwards, I set up other conjectures, using the conjectures I've proven as axioms so as to avoid an infinite regress when referring to the validity of the basic theorems on which the whole proof is based. I do not approach the Goldbach conjecture directly as it is stated. Instead of considering all prime numbers, I consider only odd prime numbers and the even numbers that are left out by disregarding the even prime 2, is expressed directly. Hence, proving the Goldbach conjecture as a whole.

After each conjecture is proven, another conjecture that leads to the algebraic interpretation of the Goldbach conjecture is set up. By following this process, I was able to reach a point where one of the conjectures I set up resembled the strong Goldbach conjecture. In the process, one of the results obtained was also used to show that the weak Goldbach conjecture is valid if the strong Goldbach conjecture is proven to be true. Assuming my prior conjectures to be true, which they were as they were mathematically proven, I was able to prove that the Goldbach conjecture is true for all even numbers regardless of their magnitude.

## Introduction

It was June 7th 1742. Christian Goldbach (1690 - 1764) wrote a letter to Euler suggesting in particular that every number is the sum of three primes (Wells). In another letter from Euler dated 30th June, 1742 (Goldbach's Conjecture - Wikipedia, the free encyclopedia) Goldbach was reminded of his original conjecture which was derived from the following statement:

Every even integer greater than 2 can be written as the sum of two primes

In this letter, Euler expressed his belief in these statements, and he regarded the statement 'every integer is a sum of two primes' a completely certain theorem (Goldbach's Conjecture - Wikipedia, the free encyclopedia). Though, he said he couldn't prove it (Wang).

Today, Goldbach's conjecture has two forms: the strong Goldbach conjecture and the weak Goldbach conjecture. Due to a number of changes in the number systems (such as the exclusion of the number 1 from the list of prime numbers), the Goldbach conjecture was modified to these forms:

Every even integer greater than or equal to 6 is the sum of two odd primes. (Wang)

Every odd integer greater than or equal to 9 can be represented as the sum of three odd primes. (Wang)

The weak Goldbach conjecture (B) can be derived from the strong one (A).

When initially stating his conjecture, Goldbach had included the even number 2 because he considered 1 to be a prime number, 'a convention

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that is no longer followed' (Weisstein). Euler then re-stated an equivalent form of the conjecture which asserted that 'all positive integers greater than or equal to 4 can be expressed as the sum of two primes' (Weisstein). It was from this expression that conjecture A above was formulated. As 3 is the smallest odd prime number, the smallest even number that can be got from the sum of the smallest odd prime numbers is 6. So, it follows from Euler's polished version of the Goldbach conjecture that all even number greater than or equal to 6 can be expressed as the sum of two odd prime numbers. Similarly, 3 being the smallest odd prime number, the smallest sum of three odd prime numbers that can be obtained is 9. This gives birth to the weak conjecture that all odd numbers greater than or equal to 9 can be expressed as the sum of three odd prime numbers. Hence, if the strong conjecture can be proved, then the weak conjecture can also be considered proven as it is a direct derivation of the strong conjecture.

As of November 6, 2010, the conjecture was tested for even numbers up to (Silva), which is a quite large number, however, not enough to prove the conjecture. This conjecture can only be considered proven if a general solution is found and not till a specific number.

The failure of the great mathematicians to prove this conjecture from the time it was formulated, including the likes of Euler and Ramanujan, despite the drastic advancement in computer technology is quite intriguing. This is why I chose to undertake this particular conjecture for my extended essay.

## **Prime Numbers**

Prime numbers are those numbers that have only two factors - 1 and the number itself.

In other words,  $n$  is a prime number iff the factors of  $n$  are 1 and  $n$ .

Numbers that are not prime are composite. So, numbers with more than two factors are said to be composite numbers.

### **Examples of prime numbers:**

2 (the only two real numbers that can be multiplied to get 2 are 1 and 2)

3 (the only two real numbers that can be multiplied to get 3 are 1 and 3)

11 (the only two real numbers that can be multiplied to get 11 are 1 and 11)

12 is a composite number as it can be expressed as the product of 12 and 1, or 6 and 2, or 3 and 4.

Previously, 1 was regarded as a prime number. However, of recent, it has been disregarded as such since it does not have two distinct factors. The only factor is 1 - the number itself.

### **Are there even prime numbers?**

An even number is one that is completely divisible by 2. That is,  $N$  is prime iff  $N \% 2 = 0$ . This implies that 2 is a factor of an even number. As said above, a prime number can have only two factors - 1 and itself. Therefore, an even number cannot be a prime number unless the two factors of the prime number are 1 and 2 in which case the number will exhibit properties of both

a prime number and an even number. In this case, the number will qualify to be both even and prime. And the only such number is 2.

In conclusion, the only even prime number is 2, as any other even number would have a factor of 2 in addition to 1 and itself, which will make it a composite number.

Therefore, all prime numbers greater than two will be odd:

Where represents odd prime numbers,

represents odd numbers.

## **Previous Attempts to Prove the Conjecture**

The first great achievements on the study of Goldbach problem were obtained in the 1920s (Wang).

The Sieve of Eratosthenes is an ancient method of getting all the prime numbers up to a certain number.

In the year 1919, the Norwegian mathematician Viggo Brun, after developing his own new type of sieve that we know today as 'Brun's sieve', conjectured that 'every large even number is the sum of two numbers each having at most nine prime factors' (Wang).

In 1923, two British mathematicians by the names of John Edensor Littlewood and Godfrey Harold Hardy, used the 'circle method', which is most frequently used in analytic number theory, proved that 'every sufficiently large odd integer is the sum of three odd primes and almost all

even integers are sums of two primes if the grand Riemann hypothesis is assumed to be true' (Wang). However, this was not considered 'proof' of the Goldbach conjecture since it was based on the grand Riemann hypothesis which was not proven.

In 1930, the Russian mathematician Lew Genrichowitsch Schnirelman used Brun's method along with his own idea of the density of integer sequence established a theorem in additive prime number theory that stated that 'Every integer greater than or equal to 2 is the sum of at most 20 primes'. (Shnirelman biography)

In 1937, Russian mathematician I. M. Vinogradov used the circle method of Hardy and Littlewood with his own method of the estimation of the exponential sum with a prime variable, to remove the dependence of the Hardy and Littlewood conjecture on the grand Riemann hypothesis.

After a series of improvements on Brun's method, in 1966, Chen Jing Run, a Chinese mathematician established that 'every large even integer is the sum of a prime and a product of at most two primes.'

All these methods were direct or indirect attempts to prove the Goldbach conjecture. Each improvement to a previous attempt brought the world closer to a proof of the apparently obvious conjecture. Each new method raised the hopes of mathematicians. The British publisher, Tony Faber went as far as to put up a one million dollar reward for the person that would submit a proof before April, 2002. Many proofs were submitted, each tackling the Goldbach conjecture from its own angle but unfortunately, none were accepted and the million dollar price went unclaimed.

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To date, the conjecture has been seen to hold until specific even integers, by means of computers. But there isn't any formal proof of the conjecture. In 1998, Joerg Richstein verified Goldbach's conjecture up to . The weaker conjecture, that every odd number can be expressed as the sum of three prime numbers was checked up to .

## **Probabilistic Approaches to the Goldbach Conjecture**

A heuristic approach involves the use of the probabilistic distribution of prime numbers to provide evidence for the conjecture.

The larger an integer becomes, the more ways there are for that integer to be represented as the sum of two other numbers.

For instance,

$$6 = 1 + 5 = 2 + 4 = 3 + 3$$

$$10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6 = 5 + 5$$

$$20 = 1 + 19 = 2 + 18 = 3 + 17 = 4 + 16 = 5 + 15 = 6 + 14 = 7 + 13 = 8 +$$

$$12 = 9 + 11 = 10 + 10$$

Effectively, the probability that one of these combinations consists entirely of prime numbers increases.

The prime number theorem (PNT) asserts that ' the number of primes is, and so the proportion of numbers that are prime is ' (Wells). Let be the number of primes greater than two and less than n. Then, by the prime number theorem:



Therefore, the probability of a number  $n$  being prime,  $P(n)$  is:

So, if  $n$  is a large even integer and  $m$  is a number between 3 and  $n$ , then the probability of  $m$  and  $n-m$  being prime simultaneously is:

However, it is assumed that the events  $m$  and  $n-m$  are statistically independent of each other, which is not the case. Consider, for example, a case in which  $n$  is divisible by three. Then  $m$  would be co-prime to 3 (i. e. the highest common factor of  $m$  and 3 would be 1) and more likely to be prime than any other number.

However, if this justification is assumed to be true, then the total number of ways to write a large even integer  $n$ , as the sum of two odd primes can be approximated as follows:

Since this quantity tends to infinity as  $n$  tends to infinity, the total number of ways in which a large even number can be written as the sum of two odd primes increases.

However, we cannot define how 'large enough' the even integer  $n$  should be before we are certain that it has at least one representation as the sum of two odd primes.

In addition, we are looking at probability of one of the representations consisting entirely of prime numbers. This implies that however probable it may be, there is always a probability, however minute, that the large even number cannot be represented as the sum of two odd primes. Therefore, we cannot be certainly sure that every even number can definitely be

represented as the sum of two odd primes.

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This can also be explored using the 'Goldbach comet'.

Mark Herkommer[1]uses an alternate probabilistic approach to the Goldbach Conjecture which he posted on his research page. It involves the partitioning of even numbers and finding the probability that there is at least one row in the partition that has both numbers as prime numbers, as illustrated below:

Consider the even number 20. This even number can be partitioned in terms of two odd numbers that sum up to it as follows:

1

19

3

17

5

15

7

13

9

11

The prime numbers in the table above are shaded.

There are two rows that have both numbers as primes (i. e. two rows where both cells are shaded). Provided that there is at least one such row, the even number that is being partitioned can be written as the sum of two prime numbers. If there was any one number that didn't have any row in the partition table similar to the one shown above with both numbers as prime numbers, then Goldbach's conjecture would be false.

Mark Herkommer then uses the density of primes to show that finding a partition of any even number that consists of both numbers being prime is ' "virtually" certain'.

He considers that partitioning of the number 100 in intervals of 10 to calculate the probability that there will be a representation which has both numbers being prime.

Interval

Number of prime numbers

Percentage of prime numbers in the odd numbers

Matching Interval

Number of prime numbers

Percentage of prime numbers in the odd numbers

Probability of having a representation with two primes

Probability of having no representation with two primes

0 - 10

3

0.6

90 - 100

1

0.2

0.12

0.88

10 - 20

4

0.8

80 - 90

2

0.4

0.32

0.68

20 - 30

2

0.4

70 - 80

3

0.6

0.24

0.76

30 - 40

2

0.4

60 - 70

2

0.4

0.16

0.84

40 - 50

3

0. 6

50 - 60

2

0. 4

0. 24

0. 76

Note:

The matching interval in the table is such that out of the two numbers that add up to 100, if one number comes from the interval then the other should come from the corresponding matching interval. E. g. if  $33 + 67 = 100$  is considered, then if 33 comes from the interval 30 - 40, the matching interval should have 67 i. e. (60 - 70).

Each interval and matching interval has 5 odd numbers. E. g. between 0 and 10 there are five odd numbers i. e. 1, 3, 5, 7, 9. The same applies to all the other intervals.

The probability of having a representation with two primes is the product of the percentage of prime numbers in an interval and the percentage of prime numbers in the matching interval. E. g. The probability of having a representation with two primes for the number 100 with one of the prime numbers being between 0 and 10, and the other being between 90 and 100 is

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Since the probability of both numbers being prime added to the probability of both numbers not being prime (i. e. either one is prime or none are prime) should add up to 1, the probability of having no representation with two primes is  $1 - [\text{probability of having a representation with two primes}]$ .

If the number 100 cannot be represented as the sum of two primes, then none of the representations from the table above, with a number from an interval and another from the matching interval, consists of two primes.

Therefore, the probability that the number 100 cannot be represented as the sum of two primes is the product of the probabilities of having no representation with two primes for all intervals. i. e. The probability that 100 cannot be represented as the sum of two primes = .

Therefore, the probability that 100 can be represented as the sum of two primes is

Mark Herkommer extended this type of investigation to different large even integers and obtained the following results:

Number, n

Probability that n can be represented as the sum of two prime numbers

1000

0. 996208045988

2000

0. 999838315754

3000

0. 999999069064

4000

0. 999999693974

5000

0. 999999983603

6000

0. 999999999995

7000

0. 999999999875

8000

0. 999999999978

9000

1. 000000000000

10000

1. 000000000000



Although the 1. 000... in the table above is a fraction less than 1, the probability clearly converges towards 1. But since the probability isn't exactly 1, we still cannot consider this proof of the Goldbach conjecture. This method, just like the previous one, only tells us that the Goldbach conjecture is 'likely' to be true.

## Approaching the Goldbach Conjecture

In order to prove the Goldbach conjecture algebraically, it is necessary to set up a few conjectures and prove them such that we can be certain about the truth of the axioms that the proof will be based on.

Conjecture 1: Every positive even number,  $E$ , can be expressed as the sum of two odd numbers.

A positive odd number,  $o$ , is of the form  $2k + 1$ .

A positive even number,  $e$ , is of the form  $2k$ .

The statements (i) and (ii) above are considered to be true. Any attempts to prove them would be trivial.

Let  $M$  and  $N$  be two odd numbers such that  $M = 2k + 1$  and  $N = 2l + 1$ . In order to prove this conjecture, the sum of  $M$  and  $N$  should simply be of the form of a positive odd number as shown in statement (ii) above (i. e.  $2k + 2l + 2$ ).

where

Since  $2k + 2l + 2 = 2(k + l + 1)$ ,

The result obtained for the sum of M and N is clearly similar to that shown in statement (ii). Therefore, from (ii), is an even number, and in general terms, it can be written as follows:

where are two positive odd numbers greater than or equal to 1.

E is a positive even number greater than or equal to 2.

Conclusion: Every positive even number can be expressed as the sum of two positive odd numbers.

Conjecture 2: The difference between any two different positive odd numbers is always a positive even number.

Again, let and be two odd numbers such that  $M =$  and  $N =$  . If both M and N are equal, then the difference between the two will be 0. If one is larger than the other, say M is larger than N, then subtracting M from N would yield a negative answer. Therefore, to get a positive difference, the smaller positive odd number, N, is subtracted from the larger positive odd number.

Since and and , can take on any value apart from 0.

Assuming that when two positive odd integers are subtracted as above, then always, the range of can be derived:

In other words if  $M > N$ , then  $m > n$ .

Going back to the difference between and ,

where

Assuming  $n$ , then as seen above,

provided that

Therefore, from (ii),  $n$  is a positive even number if  $n - p$  yields which is exactly the representation of an even number. In general, this can be written as:

where  $a$  and  $b$  are two positive odd numbers.  $E$  is any positive even number.

Conclusion: When a smaller positive odd number is subtracted from a larger positive odd number, the result is a positive even number.

Therefore, it can be seen that the difference between a positive odd number and an odd prime number is always a positive even number because odd prime numbers are also odd numbers.

since  $n > p$ .

For the equation to be valid for this situation,  $n - p > 0$ . And since the smallest odd prime number is 3, then  $n > 3$ .

Therefore, given that  $n > p$  and  $n - p > 0$ .

At this point, we have obtained the general representation of an odd number greater than or equal to 5 as the sum of an odd prime number and a positive even number. It has also been previously proven that each positive even number can be written as the sum of two positive odd numbers. Therefore, the two results can be combined to derive a general representation of an odd number greater than or equal to five as the sum of an odd prime number and two positive odd numbers, as follows:

From proof of conjecture 2, .

And from conjecture 1, .

Therefore, the result obtained from proving conjecture 2 can be re-written as follows:

The weak Goldbach conjecture asserts that every odd number greater than or equal to 9 can be written as the sum of three odd primes. Therefore, the above statement would pass as proof of the weak Goldbach conjecture if  $a$ ,  $b$ , and  $c$  were odd prime numbers.

But since all odd prime numbers are odd numbers, and the smallest odd number is 3, one can re-write the equation above as:

However, this would pass for the first odd number in the range (i. e. 9) as the smallest odd prime is 3:

However the initial expression, that includes the representation of an odd number as the sum of an odd prime and two other odd numbers, allows for the use of all odd numbers. Since odd prime numbers are just a subset of odd numbers, the odd numbers that can be used in the derived expression are limited. Hence, the two statements might look equivalent but are in fact totally different.

However, the derived expression can be counted valid if it can be proven that:

But since it is known from the proof of conjecture 1,

Therefore, the derived expression can be counted valid if it can be shown that:

## 2

Incidentally, this is the strong Goldbach conjecture. Therefore, it can be seen that the weak Goldbach conjecture can be proven if the strong Goldbach conjecture is proven.

Since substitution into the result of the proof of the second conjecture does not yield a proof for the Goldbach conjecture, it can be manipulated in other ways.

The proof of conjecture 3 yields the representation of an odd number in terms of the sum of two numbers among which one is an odd prime:

We know the sum of two odd numbers is an even number, and since we need two odd prime numbers in the general expression, we can add two odd numbers that are expressed in the general form above.

Conjecture 3: The sum of two odd numbers can be expressed as the sum of two odd primes and an even number.

As seen above, an odd number,  $n$ , can be written as where  $a$  and  $b$  are odd numbers and  $c$  is an even number.

Therefore, if  $a$  and  $b$  are two odd numbers, where  $a = 2k + 1$ ; and  $b = 2l + 1$

Since  $a$  and  $b$  are odd,  $a + b$  is even. And since  $c$  is even,  $a + b + c$  is even.

Since the sum of two odd numbers is an even number, let where  $k$  is even and  $l$  is odd.

Since the sum of two even numbers is also an even number, let where is even and

The equation above can then be expressed as follows:

Let be an even number such that . Since and , .

Here the Goldbach conjecture is considered proven. Since is an even number greater than or equal to 6, it is proven that any even number greater than or equal to six can be represented as the sum of two odd prime numbers. The strong Goldbach conjecture is proven and since the weak Goldbach conjecture is a direct derivation of the strong conjecture, it can also be proved.

Firstly, the above is a proof for the derived expression. The weak conjecture was seen to be true if the derived expression could be proven. Therefore, the weak Goldbach conjecture is proven.

Alternatively, it can be seen as follows. The weak Goldbach conjecture asserts that every odd number greater than or equal to nine can be represented as the sum of three odd primes.

From above,

Since and since the sum of an even number and an odd number is always an odd number, where is odd and .

Since 3 is an odd prime number, the right hand side of the equation can be re-written as

The whole equation can be re-written as:

## **Conclusion**

The strong Goldbach conjecture states that ' every even integer greater than or equal to six is the sum of two odd primes.' (Wang)

This can mathematically be written as:

Where  $n$  is even and  $p$  ; and  $q$  are two odd prime numbers.

This form is equivalent to the one that is mathematically derived on the previous page:

Now, the original Goldbach conjecture can be proven. The original conjecture is as follows:

Every even integer greater than 2 can be written as the sum of two primes.

The mathematical derivation above caters for all even integers greater than or equal to 6. So, the only even integer greater than 2 not catered for is 4.

And 4 can be expressed as  $2 + 2$ . 2 is an even prime number but the

Goldbach conjecture places no restrictions on the parity of the prime

number. In other words, the prime number need not be odd. Therefore, it is

indeed true that all even integers greater than 2 can be expressed as the sum of two prime numbers.

## **Appendix A – Terms and Symbols used**

Iff - If and only if

E - All positive even numbers.

EA, EB, EC,... - Various positive even numbers.

O - All positive odd numbers.

O1, O2, O3, ... - Various positive odd numbers.

Po - All odd prime numbers.

- Various odd prime numbers.