History of weibull distribution

In probability theory and statistics, the Weibull distribution is one of the most important continuous probability distributions. It was, first introduced by W. Weibull in 1939 when he was studying the issue of structural strength and life data analysis, and was formally named after him later in 1951. He proposed the “ chain” model to explain the structural strength. Based on the assumption that a structure is composed of several small components (n pieces) in series, we could consider the structure as being composed of an n-rings chain, the strength of which (or life) completely depends on the weakest ring’s strength (or life). In his model, with the assumption that the strength of different rings are independent and identically distributed, finding the strength distribution of the chain become the problem of finding the distribution of the weakest ring.

Due to the result of research conducted by Gnedenko (1943), no matter what the original distribution of the variable is, the asymptotic distribution of the minimum could only be three different forms. The Weibull distribution is one of them.

Since Weibull distribution is established on the weakest link model, which could sufficiently reflect the defect of material and the effects of stress concentration, it has been considered as appropriate model to describe strength of fiber material in practical application.

1. 2 Stress-Strength Analysis

Stress-Strength Analysis is the analysis of the survival of materials subjected to a random stress. Stress-Strength Analysis is a commonly used tool in reliability engineering.

The stress-strength reliability is given by

Where is the pdf of the strength distribution and is the cumulative distribution function of the stress distribution.

In this case, the data for the strength set would be actual data that is indicative of the strength of the material (i. e. maximum applied stress to cause failure), and the stress data would be actual stress data of the material under use conditions.

1. 3 Research Motivation & Target

Judging from physical of carbon fibers, microscopic flaws cause tensile failure to occur below the intrinsic strength of the fiber. Therefore, analysis of tensile failure data should provide insight into the distribution of flaws within fiber. The fiber might be viewed as a chain of interlocking links, with the flaws acting as “ weak links” in the chain. As the same model mentioned previously, the chain will fail at its weakest link. Usually tensile data do not confirm to rigid statistical distributions. Therefore, they cannot be represented by the specific fixed statistical distribution accurately. Instead, it is necessary to use a flexible distribution that enables the shape of the distribution function to be altered by the data itself. Under this situation, the Weibull distribution is one the natural choice.[]

Although based on the theoretical model, Weibull distribution should be an appropriate description strength distribution for carbon fibers. However, judging from empirical data, it always suggests that the Weibull distribution is not an appropriate model for carbon fibers. One plausible problem is validity of independency assumption in the “ chain of links” model. In that model, we assume a physical system is consisted of n identical units or items connected in series and each unit should be independently and identically distributed. However, in practice, this strict assumption cannot be guaranteed, which will cause the discrepancy of between real strength and Weibull distribution. For example, if there is a defect hole on n-th link, this defect hole will probably expend to adjacent link(n-1-th and n+1-th). That causes the problem of dependency.

Another popular plausible explanation is “ clamp effects” which is analysed in detail by Phoenix and Sexsmith[]. In building upon this work, Stoner[] developed new end-effect Weibull model, in which distinct Weibull distributions were used to characterize failures from true flaws and from artifacts in carbon fibers.

Although, to some extent, the end-effect Weibull model accurately descripts the data upon which it was based, it provokes our interests in pre-stress issue applied on carbon fiber before it got tested. Because in practice, such as in the strength test, fiber always suffer stress before they can be tested. For example, in the popular experiment proposed by Bader & Pries[], there is no guaranty that single fiber could be distinguished without any slight press of damage. Thus, in that experiment, they stress the fibers before they get tested, which will cause the strength test result representing all fibers performance survived from pre-stress.

This question arise our interest in study of pre-stress Weibull distribution. In the problem we will discuss in this thesis, we assume that original strength of fiber is Weibull distributed. Therefore, if there is no pre-stress applied on fiber material, the final strength will be also Weibull distributed without any changes from origins. However, judging from common sense, pre-stress could not be avoided exclusively before the strength test. They usually could occur in shipping procedure, and pre-test preparation procedure. Our goal is to find out under what conditions these “ pre-stressed” fiber have a (approximately) Weibull distribution.

In the second chapter, the results of survival fiber strength distribution under multi-type pre-stressed condition have been given out. At same time, a minimum censoring proportion has been set up to assure the pre-stress has a significant effect on original fiber material.

In the third chapter, MSE is used to measure the fitness of the pre-stressed censored sample and nearest Weibull distribution. Moreover, we discuss the goodness of fit test applied to the pre-stressed censored sample and Weibull distribution with parameters value equal to MLEs from censored sample, which is considered as nearest Weibull distribution based on the censored sample.

In the fourth chapter, the simulation results of two methods proposed in the third chapter have been discussed and analyzed.

Finally, we will give out a conclusion about survival distribution of pre-stress Weibull distribution.

Chapter 2

Weibull Family and Pre-Stressed Censored Sample

2. 1 Basic Properties of Weibull Distribution

The probability density function of a Weibull random variable is

The parameter is the shape parameter, is the scale parameter, and is the location parameter of the distribution. When , this reduces to the usual two-parameter Weibull distribution.

The Weibull distribution is related to many other probability distributions; in particular, it interpolates between the exponential distribution () and the Rayleigh distribution ().

The CDF

The cumulative distribution function for the two parameter Weibull distribution is

for , and for .

The failure rate (or hazard rate) is given by:

The Mean

The mean, , of Weibull pdf is given by:

Where is the gamma function evaluated at the value of .

The Median

The median, , is given by:

The Mode

The mode, , is given by:

The Standard Deviation

The standard deviation, is given by:

Applications

The Weibull distribution has multiple applications in practical world.

Survival analysis

Reliability engineering

Weather forecasting

General insurance

2. 2 Censored Weibull Sample

Because our primary interest is to study strength distribution of fiber material after pre-stressed, we need to generate various censored Weibull sample which is applied multi-type pre-stress.

Given and are independent random variables where represent original strength of material and pre-strength individually. So the problem of our interest is what distribution does variable have? To state the problem more clearly, we assume the original strength of material always yield to Weibull distribution family. And the pre-stress yield to three different distribution families which are Weibull, Normal and Gamma. Based on different parameter choice, we will try to find out what values of the parameters give us that survival strength yield to or approximately yield to Weibull distribution.

Mentioned in Chapter 1, we assume is weibull distributed with different shape and scale parameter. In this paper, we will from (pre-stress variable) from three different distribution: Weibull distribution, Gamma distribution, and Normal distribution. Also as similar as original strength set, in each distribution option, different parameters are chosen to generate various censored Weibull sample.

Meanwhile, since trivial results are not what we expected, a certain censoring ratio has been set up to guarantee there is a significant effect of pre-stress applied on original fiber carbon material. Serving to this purpose, samples with censoring proportion is 0. 5 or greater than 0. 5 will be kept, which is considered as plausible limit of censoring ratio.

Chapter 3 Mean Squared Error and Goodness of Fit Test

The Mean Squared Error (MSE) is a measure of how close fitted curve is to data points. For every data point, the vertical distance from the point to the corresponding value on the fitted curve (the error) will be taken, and the value will be squared. Then those squared values have been added up for all data points, and been divided by the number of points, which is considered as a mean. Since all errors have been switched into positive values on matter what original sign they have, negative values do not cancel positive values. The smaller the Mean Squared Error, the closer the fit curve is to the data points.

MSE has been widely used for quantitative performance metric in the field of statistical regression and engineering, such as signal processing.

3. 1 Comparison of MSE of Censored Weibull Sample

Based on the censored Weibull Sample we got, we will calculate out MLE of parameters for Weibull distribution, noted as and , which is considered as nearest Weibull distribution to censored weibull sample. Since our objective is to measure fitness of censored Weibull sample and nearest Weibull distribution, MSE between of censored Weibull sample and Weibull distribution with parameters and . We establish the MSE as following way:

Where are censored Weibull sample which is sorted as . is empirical CDF at sample point . is defined as sample size. And is CDF of Weibull distribution with parameter , which is MLE's got based on censored Weibull sample.

In addition, a baseline of MSE comparison is established by calculating MSE between the sample of size from Weibull() and Weibull() distribution.

Where consist the sample of size from Weibull(), which is sorted as ascending order: . is empirical CDF at sample point . And is CDF of Weibull distribution with parameter .

3. 2 Comparison via Simulation Result

We conducted a large-scale simulation study to compare of the performance of difference between base MSE and sample MSE depends on different censoring stress distribution and parameter chosen. For example, let original strength distribution yields to Weibull(1, 1), and censoring stress distribution is Gamma distribution with parameter value chosen from the range 0. 25 to 2. 5, then we will focus on the performance of fitness of cersored sample to Weibull distribution from various aspects such as censoring proportion, shape parameter value of censoring distribution and scale parameter value of censoring distribution.

From above figure, we could not hardly see that the MSE increases with increasing of cersoring proportion, which could be simply interpreted as survival data will go off the original strength distribution when considerable proportion of original strength failed the pre-stress test. Meanwhile, the base MSE keeps relatively stable, since it will not be affected by different pre-stress distribution. Therefore, it is obvious to tell that difference between MSE and base MSE is raised up when the censoring proportion increases, which indicates fitness will go worse at same time.

The series of comparison MSE chart are arranged by the ascending order of value of Shape Parameter of Gamma Distribution. If we focus on the changing trend of Average Difference between Sample MSE and Base MSE, we could easily figure out that the Average Difference decreases as long as value of shape parameter become smaller and smaller. This shows that fitness of sample is improved by using smaller shape parameter of pre-stress Gamma distribution.

On the other side, if we focus on each individual chart, it is not too hard to find that when the value of Scale increases, difference between Sample MSE and Baseline decreases, which indicates fitness is improved by larger scale parameter chosen. This phenomenon could be interpreted as larger scale brings in less spread compared with smaller ones create absolute peak in pdf, which breaks the continuity of censored sample.

Above figure is overall behavior of MSE based on different choice of shape and scale parameter of pre-stress gamma distribution. To summary up, In general MSE boosts up where is smaller scale value and larger shape values. Fitness performance indicates censored sample is still Weibull (or approximately Weibull) distributed when shape and scale parameter choice close to minimum and maximum individually.

Chapter 4 Fit Study Based on Weibull Goodness-of-Fit Tests

In order to study whether the censored weibull distribution data yields to a weibull distribution, we first generate different censored weibull samples based on different censoring function. Then we use the weibull distribution with parameters value equal to MLE of censored sample as hypothesis. Three Goodness-of-Fit Tests are manipulated to censored weibull samples. We check whether simulation results shows censored weibull sample still yields to a weibull distribution.

Goodness-of-fit tests

Goodness-of-fit tests for the two-parameter Weibull distributions have drawn considerable attention since its critical importance. Mann and others (1973), Smith and Bain (1976), Stephens (1977), Littell and others (1979), Chandra and others (1981), Tiku and Singh (1981), Wozniak and Warren (1984), and James and others (1989) have gave out universal discussion of this problem. Mann and others (1973) and Tiku and Singh (1981) proposed new statistics to test the goodness-of-fit of two-parameter Weibull distribution.), Smith and Bain (1976) proposed a test statistic to test normality which is analogous to the Shapiro-Francis statistic. The Smith and Bain statistic was derived from the sample correlation between the order statistics of a sample and the expected value of the order statistics under the assumption that the sample comes from a two-parameter Weibull distribution. They provided critical values for samples containing 8, 20, 40, 60, 80 observations. Stephens (1977) provided tables of the asymptotic critical values of the Anderson-Darling statistic and the Cramer-von Mises statistic for various significance levels. Littell and others (1979) made a comparison among the Mann, Scheuer, and Fertig statistic, the Smith and Bain statistic, the modified Kolmogorov-Smirnov statistic, the modified Anderson-Darling statistic, and the modified Cramer-von Mises statistics through a various of power studies for sample size n = 10 to 40. Critical values for the , , and statistics for n = 10 to 40 have been calculated and provided at the same time. Chandra and others (1981) calculated critical values for the Kolmogorov-Smirnov statistic for n = 10, 20, 50 and infinity for three situations. James and others produced extensive tables of goodness-of-fit critical values for the two parameter Weibull distributions developed through simulation for the Kolmogorov-Smirnov statistic, the Anderson-Darling statistic, and Shapiro-Wilk-type correlation statistics.

Kolmogorov-Smirnov statistic

Kolmogorov-Smirnov test (Chakravart, Laha, and Roy, 1967) is widely used for comparing a sample with a reference probability distribution (one-sample K-S test), or for comparing two samples (two-sample K-S test). Since it is highly sensitive to distinguish the difference of the empirical cumulative distribution functions of the tested samples, it has been considered as one of the most useful nonparametric methods for comparing two samples. Kolomogorov-Smirnov test is based on the distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution, or between the two samples' empirical distribution functions.

Kolmogorov-Smirnov test has been modified to serve as a goodness of fit test here. The modified Komogorov-Smirnov D statistic is given by

where is the empirical distribution function of the sample and is the fitted distribution.

Anderson-Darling statistic

Anderson-Darling test (Stephens, 1974) is used to test whether the data follow a particular distribution. It is named after Theodore Wilbur Anderson and Donald A. Darling who proposed it in 1952. Anderson-Darling test is based on Kolmogorov-Smirnov test and enjoys superior property of more sensitive due to specific distribution application in calculating critical values. Contrarily, its disadvantage is that critical values must be calculated for each distribution.

Anderson-Darling test has been modified to serve as a goodness of fit test here. The modified Anderson-Darling statistic is given by

Where and is the th-order statistic.

Shapiro-Wilk-Type correlation statistics

In statistics, the Shapiro-Wilk test is used as the tester of goodness-of-fit test of normal distribution. It has the superior power compared with other statistics in detecting the data comes from a relatively wide range of other distributions for testing goodness-of-fit of normal distributions which has been proposed from Monte-Carlo study of Shapiro and others (1968).

Shapiro-Wilk test has been modified to serve as a goodness of fit test here. The statistic first proposed by Shapiro and Francia (1972) but with approximate " scores" suggested by Filliben (1975) and modified form fitting for good-of-fit test for Weibull distributions is given by:

And

Is the median score, in the spirit of Filliben, except that these scores depend upon the maximum likelihood estimate the Weibull shape parameter .

If a variable has the two-parameter Weibull distribution, the variable has an extreme value distribution. Calculating goodness-of-fit on this scale has advantages. Since the extreme value distributions are defined by location and scale parameters, the critical values for the correlation statistic are not dependent on the true shape parameter. Thus, for two-parameter Weibull distribution, we propose the correlation-type statistic , where

And

Simulation Result