Laplace transforms

Laplace Transforms – Motivation

convenience

– differential eqns become algebraic eqns.

– easy to handle time delays

– frequency response analysis to determine how the system responds to oscillating inputs

Block Diagram Algebra

– doing math with pictures

– arithmetic for manipulating dynamic components using boxes and arrows

Laplace Transform – Review

Given a function f(t)

Notes –

f(t) – defined for t from 0 to infinity

f(t) – suitably “ well-behaved”

– piecewise continuous, integrable

Linearity of Laplace Transforms

the Laplace transform is a linear operation

we will use Laplace transforms to analyze linear dynamic systems

if our models aren’t linear, then we will linearize

Useful Laplace Transforms for Process Control

We need a small library of Laplace transforms for

– differentiation

– step input

– pulse/impulse functions

– exponentials

– oscillating functions

because these are common functions that we will encounter in our equations

Let’s think about a simple linear differential equation example:

with V and F as constants

Library of Useful Transforms

differentiation

– initial conditions disappear if we use deviation variables that are zero at an in initial steady state

unit step function (Heaviside fn.)

Library of Transforms

exponential

– exponentials appear in solutions of differential equations

» a provides information about the speed of the response when the input changes. If a is a large negative number, the exponential decays to zero quickly

» What happens if a is positive?

– After we have done some algebra to find a solution to our ODEs in the Laplace domain, we must invert the Laplace transform if we want to get a solution in the time domain. We sometimes use partial fraction expansion to express the Laplace expressions in a form that can be easily inverted.

CSTR Example – Transform Model (in deviation variables) using our library of transforms, the Laplace transform of the model is: For a step change in feed concentration at time zero starting from steady state. Tank Example – Solution

Solve for CA(s)

If we like, we can rearrange to the form: This is the solution in the Laplace domain. To find the solution in the time domain, we must invert the Laplace transforms

CSTR Example Solution

inverse Laplace transform

– Can be determined using a complex integral

easiest approach is “ table lookup”

Use Table 4-1, entry 5

Maple is good at inverting Laplace transforms too

The Impulse Function

limit of the pulse function (with unit area) as the width goes to zero and height becomes infinite transform

CSTR – Impulse Response physically – dump some pure A into reactor, all at once input function Transform

time response

Interpretation of Impulse Response dump a bag of reactant into the reactor in a very very short time we see an instantaneous jump to a new concentration due to the impulse input

concentration then decays back to the original steady-state concentration

Time-Shifted Functions – Representation of Delays Laplace transform for function with time delay

Just pre-multiply by an exponential.

How could we prove this?

– change of variables in integration in expression for Laplace Transform (see p. 103 of Marlin, p. 115 in first ed.)

Reactor Example with Time Delay

Suppose we add a long length of pipe to feed…

– assume plug flow

– It will take a time period, q minutes, before the change in

concentration reaches the tank, and begins to influence cA

– delay differential equation

» difficult to solve directly in time domain

» easy to solve with Laplace transforms

Tank Example with Time Delay – Solutionresponse to step input in cA0 time response

Final Value Theorem An easy way to find out what happens to the output variable if we wait a long time. We don’t have to invert the Laplace transform!

Why is it true?

– Consider the Laplace transform of a time derivative now let s approach zero

provided dy/dt isn’t infinite between t= 0 and t®¥ (i. e y(t) is STABLE) This will be true if Y(s) is continuous for s³0

Using the Final Value Theorem – Step Response Reactor example – final value after a step input

What can we do with Laplace Transforms so far.

Take Laplace transforms of linear ODEs (in deviation variables).

Substitute Laplace transform expressions for different kinds of inputs we are interested in:

– Steps, pulses, impulses (even with dead time)

Solve for the output variable in terms of s.

Invert the Laplace transform using Table 4. 1 to get the solution in the time domain.

Find the final steady state value of the output variable, for a particular input change, even without inverting the Laplace transform.

Laplace transforms are mostly used by control engineers who want to determine and analyze transfer functions.

compact way of expressing process dynamics

relates input to output

p(s), q(s) – polynomials in s

– q(s) will also contain exponentials if time delay is present

Once we know the transfer function of the process, we can use it to find out how the process responds to different types of input changes: