Vector theorems - math problem example

Science, Mathematics

Vector Theorems

XXXXX No. XXXXX MATHEMATICS VECTOR THEOREMS GEOMETRY XYZ 15 November, Theorem: A triangle and its medial triangle have the same centroid.

Suppose that ABC is a given triangle as shown in the figure below with vertices given as

A= (a1, a2), B= (b1, b2), and C= (c1, c2).

Figure 1

Let D, E, and F be the mid-points of sides AB, BC, and AC respectively. By joining these points, a triangle DEF is formed which is called medial triangle. This medial triangle is sometimes also known as auxiliary triangle (Dixon, 1991). The medial triangle DEF consists of the vertices

D= (d1, d2) = ((a1+ b1)/2, (a2+ b2)/2)

E= (e1, e2) = ((b1+ c1)/2, (b2+ c2)/2)

F= (f1, f2) = ((a1+ c1)/2, (a2+ c2)/2)

Now let KLM be the mid points of DE, DF, and EF respectively. By joining these points we get another triangle KLM i. e. the medial triangle of DEF. Therefore,

K= (k1, k2) = ((a1+ b1)/2+ (b1+ c1)/2), (a2+ b2)/2+ (b2+ c2)/2)

= ((a1+ 2b1+ c1)/4, (a2+2b2+ c2)/4)

Similarly, L = (l1, l2) = ((2a1+ b1+ c1)/4, (2a2+b2+ c2)/4), and

M = (m1, m2) = ((a1+ b1+ 2c1)/4, (a2+b2+ 2c2)/4)

Since we have a medial triangle DEF as shown above, for convenience we draw the same triangle separately as shown below. Here D (a, 0, 0), E (0, b, 0), and F (0, 0, c) be the vertices of medial triangle while N is the centroid of the triangle.

By distance formula, N has the coordinates as (a/3, b/3, c/3).

Now DN = $\sqrt{}$ [(a/3-a)^2 + (b/3-0)^2 + (c/3-0)^2]

= $\sqrt{}$ (4a^2/9) + (b^2/9) + (c^2/9)

= $\sqrt{}$ [(4a^2 + b^2 + c^2)]/3

3DN = $\sqrt{}$ [(4a^2 + b^2 + c^2)] (1)

Let D′ be the mid-point of EF, therefore its coordinates will be (0, b/2, c/2).

Now we find the distance between N and D′.

D′N = $\sqrt{}$ [(a/3-0)^2 + (b/3-b/2)^2 + (c/3-c/2)^2

= $\sqrt{}$ (a^2/9) + {(b^2/9 + b^2/4 – 2 (b/3) (b/2)} + {(c^2/9 + c^2/4 – 2 (c/3) (c/2)}

= $\sqrt{}$ (a^2/9) + (b^2/36) + (c^2/36)

= $\sqrt{}$ [(4a^2 + b^2 + c^2)]/6

6 D′N = 3 DN as by using equation (1)

2 D′N = DN

Similarly 2 E′N = EN, and 2 F′N = FN.

Thus, it means that the centroid is located two thirds of the way from the original vertex to the midpoint of the opposite side of the triangle.

Now, since we have a triangle ABC and E is the middle point of BC, and P is the centroid of ABC. Therefore, by using above theorem we have OP = OA + 2/3 AE. Now we will find the centroid of ABC.

Let P = centroid of ABC = OP = OA + 2/3 AE

Thus P = (a1+ (2/3) [e1- a1], a2+ (2/3) [e2- a2])

Then P = (a1+ (2/3) [(b1+ c1)/2- a1], a2+ (2/3) [(b2+ c2)/2- a2])

P = (a1+ (b1+ c1)/3-(2/3) a1, a2+ (b2+ c2)/3-(2/3) a2)

P= ((a1+ b1+ c1)/3, (a2+ b2+ c2)/3) (2)

Now we will find the centroid of DEF

Let Q = centroid of DEF = OQ = OD + 2/3 DM

Then Q = (d1+ (2/3) [m1- d1], d2+ (2/3) [m2- d2])

Then Q = (d1+ (2/3) ((a1+ b1+ 2c1)/4-(a2+ b2)/2), d2+ (2/3) ((a2+ b2+2c2)/4-(a2+ b2)/2))

Q = (d1+ (2/3) ((-a1- b1+ 2c1)/4), d2+ (2/3) ((-a2- b2+2c1)/4))

Q= ((a1+b1)/2+ (-a1- b1+2c1)/6, (a2+ b2)/2+ (-a1- b1+2c1)/6)

Q= ((2a1+2b1+2c1)/6, (2a2+2b2+2c2)/6)

Q= ((a1+ b1+ c1)/3, (a2+ b2+ c2)/3) (3)

From above equation no. 2 and equation no. 3 it is evident that P= Q. Thus it has been proved that the centroid of the triangle ABC is equal to the centroid of medial triangle DEF.

Reference

Dixon, R. (1991). Mathographics, New York: Dover, p. 56