Circumscribable quadrilaterals essay

CHAPTER 1 INTRODUCTION 1. 1 Introduction Geometry is one of the most interesting fields of mathematics. From the ancient times of the Greeks up to now, it has held captive the imagination of many mathematicians, artists, scientists, engineers and architects. Its application to modernization and technological advancement cannot be denied. Thus, it must be given emphasis in educational institutions particularly in secondary schools. The low achievement test results in mathematics of high school students in the Philippines have always been the subject of study in researches and the center of discussion in conferences.

Different strategies and recourses have been drafted to be implemented in schools towards a better performance in the said subject. One such strategy is exposure to varied problems and concepts that cater to higher order thinking skills of students for them to develop critical and analytical skills. The quadrilateral is a very interesting concept for it is rich in content and applications. Quadrilaterals such as the golden rectangle and cyclic quadrilateral are famous for their intricate properties.

In high schools, however, this concept is not given emphasis and teachers are confined within the types of quadrilaterals enumerated in the textbook. In effect, students' learning is shallow. Their interests are not stimulated and they think quadrilaterals are very simple. But when given problems on quadrilaterals that are of moderate difficulty, they are stumped. 1. 2 Statement of the Problem The aims of this paper are to define circumscribable quadrilaterals and to present the proofs of its properties. It also intends to discuss and prove the conditions for a quadrilateral to be circumscribable.

This paper is inspired by Charles Worrall's article entitled " A Journey with Circumscribable Quadrilaterals" published in the Delving Deeper section of the Mathematics Teacher in October 2004. In the paper, the author explained the importance of inquiry in teaching and learning mathematics. He also emphasized creative thinking in order to better understand and appreciate the beauty of mathematics particularly geometry. 1. 3 Scope and Limitation of the Study Due to the limited time devoted in making this paper, it is confined only to the definition of circumscribable quadrilaterals, its properties, and the conditions for its existence.

Also, we assume that all quadrilaterals mentioned in this paper are convex quadrilaterals. Furthermore, proofs are done within the confines of Euclidean geometry although projective geometry and trigonometry could also be used. Finally, all figures are constructed using the mathematics software, The Geometer's Sketchpad. CHAPTER 2 PRELIMINARIES This chapter aims to discuss the definitions, postulates, and theorems needed to understand the concepts on circumscribable quadrilaterals. Illustrations and examples are also given to provide a clearer idea of the topic being discussed. 2. 1 CIRCLES

Definition 2. 1. 1. A circle is the set of all points in the plane that are a given distance from a given point. The given point is the center of the circle, and the given distance is the radius. A segment that joins the center to a point on the circle is also called radius. (The plural of radius is radii.) Illustration: A circle may be named using one capital letter that is its center. For example, a circle whose center is at P will be called circle P. Definition 2. 1. 2. A point is inside (in the interior of) a circle if its distance from the center is less than the radius. Illustration:

Point A is in the interior of circle O. Definition 2. 1. 3. A point is outside (in the exterior of) a circle if its distance from the center is greater than the radius. Illustration: Point C is in the exterior of circle O. Definition 2. 1. 4. A tangent line to a circle is a line that intersects the circle at exactly one point. This point is called the point of tangency or point of contact. Illustration: [pic] is tangent to circle O at point K. K is the point of tangency. Definition 2. 1. 5. A tangent segment is a part of a tangent line between the point of tangency and a point outside the circle.

Illustration: [pic] is a tangent segment to circle O. Definition 2. 1. 6. Tangent circles are circles that intersect each other at exactly one point. Definition 2. 1. 7. Two circles are externally tangent if the centers of the tangent circles lie on opposite sides of the tangent line. Illustration: Circles N and P are externally tangent circles. Definition 2. 1. 8. Two circles are internally tangent if one of the tangent circles lies inside the other. Illustration: Circles O and R are internally tangent circles. Theorem 2. 1. 9.

A tangent line is perpendicular to the radius drawn to the point of tangency. Illustration: Line m is perpendicular to [pic] at point A. Theorem 2. 1. 10. If a line is perpendicular to a radius at its outer endpoint, then it is tangent to the circle. Illustration: If line m is perpendicular to radius [pic] at A, then it is tangent to circle B. Theorem 2. 1. 11. (Tangent Segments Theorem) Tangent segments drawn to a circle from the same exterior point are congruent. 2. 2

TRIANGLES Definition 2. 2. 1. If A, B, and C are three noncollinear points, then the set [pic] is called a triangle.

Illustration: ABC is a triangle. Definition 2. 2. 2. The circumcircle of a triangle is the circle that contains all three of the vertices of the triangle. The center and the radius of this circumscribed circle are referred to as the circumcenter and the circumradius, respectively, of the triangle. Illustration: Circle O is the circumcircle of ? ABC with circumcenter O and circumradius OC. Theorem 2. 2. 3. The three perpendicular bisectors of the sides of a triangle are concurrent at the circumcenter of the triangle. Illustration: O is the circumcenter of ? EFG Definition 2. 2. 4.

The incircle of a triangle is the circle that is tangent to all three sides of the triangle. The center and the radius of this inscribed circle are referred to as the incenter and the inradius, respectively, of the triangle. Illustration: Circle X is the incircle of ? ABC with incenter X and inradius XY. Postulate 2. 2. 5. (The SSS Congruence Postulate)If there exists a correspondence between the vertices of two triangles such that three sides of one triangle are congruent to the corresponding sides of the other triangle, then the two triangles are congruent. Illustration: [pic] Postulate 2. . 6. (The SAS Congruence Postulate)If there exists a correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding parts of the other triangle, then the two triangles are congruent. Illustration: [pic] Postulate 2. 2. 7. (The ASA Congruence Postulate) If there exists a correspondence between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding parts of the other triangle, then the two triangles are congruent. Illustration: [pic] Theorem 2. . 8. (HyL Congruence Theorem) If there exists a correspondence between the vertices of two right triangles such that the hypotenuse and a leg are of one triangle are congruent to the corresponding parts of the other triangle, then the two right triangles are congruent. Illustration: [pic][pic] Theorem 2. 2. 9. (HyA Congruence Theorem) If there exists a correspondence between the vertices of two right triangles such that the hypotenuse and an acute angle are of one triangle are congruent to the corresponding parts of the other triangle, then the two right triangles are congruent. Illustration: pic] 2. 3 QUADRILATERALS Definition 2. 3. 1. If A, B, C and D are four points in a plane such that no three of which are collinear, then the set [pic] is called a quadrilateral. Illustration: ABCD is a quadrilateral. Definition 2. 3. 2. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. Illustration: MNOP is a parallelogram. Definition 2. 3. 3. A rectangle is a parallelogram in which at least one angle is a right angle. Illustration: EFGH is a rectangle. Definition 2. 3. 4. A rhombus is a parallelogram in which at least two consecutive sides are congruent.

Illustration: JKLM is a rhombus. Definition 2. 3. 5. A kite is a quadrilateral in which two distinct pairs of consecutive sides are congruent. Illustration: USVT is a kite. Theorem 2. 3. 6. A diagonal of a rhombus bisects opposite angles of the rhombus. Figure (a) Proof. Let ABCD be a rhombus with diagonal BD. This implies that AB = CB = AD = CD. By reflexivity, BD = BD. Thus, [pic] by SSS Congruence Postulate. Moreover, [pic]and [pic] since they are corresponding parts of congruent triangles. Therefore, we have shown that a diagonal of a rhombus bisects opposite angles of the rhombus.

Theorem 2. 3. 7. The angle bisectors of a kite and a rhombus are concurrent. Proof. We prove first the concurrency of the angle bisectors of a kite. Let ABCD be a kite such that AB = AD and BC = DC with diagonal AC . (Figure b). Figure (b) Thus we can say that ? ABC is congruent to ? ADC by SSS Congruence Postulate. It follows that [pic]BAC[pic][pic]DAC and [pic]BCA[pic][pic]DCA since they are corresponding parts of congruent triangles. This implies that AC bisects [pic]A and [pic]C. This shows that the diagonal AC is the angle bisector of [pic]A and [pic]C. Figure (c)

Let the bisector of [pic]ABC intersect AC at H and the bisector of [pic]ADC intersect AC at H’ (Figure c). Because ? ABC[pic]? ADC, then [pic]ABC[pic][pic]ADC. This implies that [pic]CBH[pic][pic]CDH’. Also [pic]and [pic]. Thus, ? BHC[pic]? DH’C by ASA Congruence Postulate. It follows that CH[pic]CH’ since they are corresponding parts of congruent triangles. Therefore, H and H’ are one and the same point on segment AC. Hence, we have shown that the angle bisectors of a kite are concurrent. (Figure d). Figure (d) Next, we prove that the angle bisectors of a rhombus are also concurrent. According to Theorem 2. 6, the diagonals of a rhombus are also its angle bisectors. Therefore, the angle bisectors of a rhombus are concurrent. In the figure below (Figure e), JL and IK are the angle bisectors of rhombus IJKL and these angle bisectors are concurrent at point M. Figure (e) 2. 4 OTHER DEFINITIONS, THEOREMS, AND POSTULATES Definition 2. 4. 1. The distance around a polygon is called the perimeter of the polygon while half of this perimeter is called its semiperimeter. Illustration: Quadrilateral EFGH has a perimeter of 13 units and its semiperimeter is 6. 5 units. Postulate 2. 4. 2. (Segment Addition Postulate)

If A, B, and C are points on a real number line and B is between A and C, then AB + BC = AC. Example 2. 4. 3. Points A, B, and C are collinear. The coordinates of A and B are – 4 and 5, respectively. If B is between A and C and AC = 14, what is the coordinate of C? Solution. Since B is between A and C, then by Segment Addition Postulate we have, AB + BC = AC. But AB = [pic] = 9 and AC = 14. Thus, by substitution, 9 + BC = 14 which yields BC = 5. Hence, C is 5 units away from B. Moreover, the three points are located on the number line as follows: [pic] Note that C must be located 5 units to the right of B since B lies between A and C.

Thus, the coordinate of C is 10. Theorem 2. 4. 4. (Point on Angle Bisector Theorem) A point is on the bisector of an angle if and only if the point is equidistant from the sides of the angle. Proof. The theorem can be written as two statements and we will prove each as follows: ([pic]) If a point is on the bisector of an angle, then it is equidistant from the sides of the angle, that is, the lengths of the perpendicular segments from the point to the sides of the angle are equal. Figure (f) Figure (g)

Let D be a point on the bisector [pic]of [pic]. Let F and G be points on the sides [pic]and [pic], respectively, such that [pic]and [pic]. (Figure f). We have to prove that[pic]. Clearly, [pic]and [pic]are right triangles. Moreover, since [pic] bisects [pic], then [pic]and by reflexivity, [pic]. Therefore, [pic][pic][pic] by HyA Congruence Theorem. Hence, [pic]since they are corresponding parts of congruent triangles. ([pic]) If the lengths of the perpendicular segments from a point in the interior of an angle to the sides of the angle are equal, then the point is on the bisector of the angle.

Let P be in the interior of [pic]such that [pic], [pic], and [pic]. (Figure g). By constructing [pic], we have formed right triangles BAP and BCP. Since [pic] and [pic], then [pic]by HyL Congruence Theorem. Since [pic][pic]and [pic] are corresponding parts of congruent triangles, then they are congruent. This implies that BP bisects[pic]. Therefore, we have shown that P is a point on the bisector of[pic]. CHAPTER 3 CIRCUMSCRIBABLE QUADRILATERALS AND THEIR PROPERTIES The concept of quadrilaterals inscribed in a circle, called cyclic quadrilaterals, has been widely known and studied.

Its properties are discussed comprehensively in most geometry books so that when related problems are given in mathematics competitions they are easily solved. On the other hand, another type of quadrilateral, those that have an inscribed circle would seem strange and vague to most high school teachers and students for its properties are not given sufficient attention even in college geometry books. We know that all triangles have an incircle, but a quadrilateral does not always have one. In this chapter, we define circumscribable quadrilaterals and present the proofs of its properties.

Definition 3. 1. A quadrilateral is said to be circumscribable if there is a circle such that each side of the quadrilateral is tangent to the circle. The figure below shows circumscribable quadrilateral ABCD with circle O tangent to all its sides. We say that quadrilateral ABCD contains circle O and circle O is said to be an inscribed circle. The center of the inscribed circle of the circumscribable quadrilateral is called the incenter and the segment joining the incenter and a point on the incircle is the inradius of the quadrilateral. Figure 3. 1. A circumscribable quadrilateral

Circumscribable quadrilaterals are also known as tangential quadrilaterals, circumscriptible or circumscribed quadrilaterals, and inscriptable quadrilaterals. Squares, rhombi, and kites are circumscribable quadrilaterals. To construct the inscribed circle of a square, a kite, and a rhombus, the steps are as follows: 1. Construct the four angle bisectors of a square, a kite, or a rhombus. Recall from Theorem 2. 27 that these angle bisectors are concurrent. Mark the point of intersection of these angle bisectors as P. (Figure 3. 2) Figure 3. 2 2. Drop a perpendicular from P to one of the sides. (Figure 3. 3).

Note that by Theorem 2. 31, the length of the segment drawn from P to the foot of the perpendicular of all the four sides of the rhombus, kite, or square are equidistant. Figure 3. 3 3. Draw the circle whose center is and whose radius is the length of the segment ([pic]) found in step 2. See Figure 3. 4. This is the desired inscribed circle. Figure 3. 4 Note, however, that not all quadrilaterals are circumscribable. For example, it is impossible to inscribe a circle in a rectangle that is not a square because if a circle is drawn inside the rectangle, it can be tangent only to at most three sides of the rectangle.

Figure 3. 5 illustrates this case. Figure 3. 5. A quadrilateral that is not circumscribable What properties must a quadrilateral possess for it to be circumscribable? This question will be answered in the next chapter. Conversely, what are the properties of circumscribable quadrilaterals? This chapter discusses thoroughly the answers to this question. Let us consider the example below to investigate circumscribable quadrilaterals. Example 3. 2. Suppose a quadrilateral with consecutive sides of lengths 25, 40, and 36 units is circumscribable. Find the length of the fourth side.

Solution. Figure 3. 6 Let the circumscribable quadrilateral be ABCD such that AB = 25, BC = 40, and CD = 36. We have to find the length of segment AD. Let P, Q, R, and S be the points of the tangency of the sides AB, BC, CD, and AD, respectively, to the inscribed circle. Let AP = x. By Segment Addition Postulate, AB = AP + PB since AB = 25 and AP = x, then by substitution, PB = 25 – x. By the Tangent Segments Theorem, PB = BQ = 25 – x. Therefore QC = 40 – (25 – x) = 15 + x. Again, by Tangent Segments Theorem, RC = QC = 15 + x, thus DR = 36 – (15 + x) = 21 – x = DS.

Since AS = AP = x and AD = AS + DS, then AD = x + (21 – x) = 21 units. We have seen in the preceding example that the sum of the lengths of one pair of opposite sides is equal to the sum of the lengths of the other pair. That is, AB + CD = 25 + 36 = 61 and BC + AD = 40 + 21 = 61. This illustrates the next theorem. Theorem 3. 3. In a circumscribable quadrilateral, the sum of the lengths of a pair of opposite sides is equal to the sum of the lengths of the other pair of opposite sides. Figure 3. 7 Proof. Let ABCD be a circumscribable quadrilateral that contains circle O (Figure 3. 7).

We want to show that AB + CD = AD + BC. Let the sides AB, BC, CD, and AD, be tangent to circle O at points P, Q, R, and S, respectively. By the Tangent Segments Theorem, AP = AS, BP = BQ, CQ = CR, and DR = DS. Let AP and AS be denoted by a, BP and BQ by b, CQ and CR by c, and DR and DS by d. Thus, AB = a + b, BC = b + c, CD = c + d, DA = d + a. It follows that AB + CD = a + b + c + d and BC + DA = b + c + a + d. Therefore, by transitivity, AB + CD = BC + DA. Example 3. 4. Given circumscribable quadrilateral JKLM and JK = 2x, KL = 3x – 2, LM = 2x + 1, and MJ = 12 units (Figure 3. 8). Find the lengths of JK, KL, and LM.

Figure 3. 8 Solution. Since JKLM is a circumscribable quadrilateral, then by Theorem 1, we can say that JK + LM = KL + MJ. By substitution, we have the equation, 2x + 2x + 1 = 3x – 2 + 12. Solving for x, we get x = 9. Therefore by substitution, JK = 18, KL = 25, and LM = 19. To check, 18 + 19 = 25 + 12 = 37. We saw earlier that a square, rhombus, and kite are circumscribable. Moreover, their four angle bisectors are concurrent. We will show in the following theorem that this is a property of any circumscribable quadrilateral. Theorem 3. 5. In a circumscribable quadrilateral, the four angle bisectors are concurrent.

Figure 3. 9. Angle bisectors in a circumscribable quadrilateral Proof. Let ABCD be a circumscribable quadrilateral with inscribed circle O. We wish to show that the bisectors of [pic], [pic],[pic], and [pic] intersect at a common point. To do this, first we draw the radii to the four sides. Since ABCD is circumscribable, then the radii drawn are perpendicular to the sides and congruent to each other. By the Point on Angle Bisector Theorem, point O lies in the angle bisector of [pic], [pic],[pic], and [pic]. Thus we have shown that the angle bisectors of a circumscribable quadrilateral are concurrent. Example 3. . Suppose that the quadrilateral PQRS inscribes a circle with center O. Prove that the sum of the angles [pic]POQ and [pic]ROS equals 180?. Figure 3. 10 Solution. By Theorem 2. 31, we know that point O is on the angle bisector of [pic]P, [pic]Q, [pic]R, and [pic]S. Considering ? POQ, [pic]POQ = 180? – [pic]QPO – [pic]OQP. But [pic]QPO = ? [pic]P, [pic]OQP = ? [pic]Q. By substitution, [pic]POQ = 180? – ? [pic]P – ? [pic]Q. Simplifying we have, [pic]POQ = 180? – ? ([pic]P + [pic]Q). (1) In ? ROS, [pic]ROS = 180? – [pic]SRO – [pic]RSO. But [pic]SRO = ? [pic]R and [pic]RSO = ? [pic]S.

By substitution and simplification we get, [pic]ROS = 180? – ? [pic]R – ? [pic]S, [pic]ROS = 180? – ? ([pic]R + [pic]S). (2) Adding (1) and (2), we have [pic]POQ + [pic]ROS = 360? – ? ([pic]P + [pic]Q + [pic]R + [pic]S). But [pic]P + [pic]Q + [pic]R + [pic]S = 360? thus, [pic]POQ + [pic]ROS = 360? – ? (360?). Therefore, [pic]POQ + [pic]ROS = 180?. Another interesting property of circumscribable quadrilaterals is presented in the next theorem. Theorem 3. 7. In a circumscribable quadrilateral, the incircles of the two triangles formed by a diagonal are tangent to each other.

Proof. Figure 3. 11 Let ABCD be a circumscribable quadrilateral such that the diagonal AC forms triangles ADC and ABC whose incircles are X and Y, respectively. We must show that circle X is tangent to circle Y at a point on AC. First, we consider circles X and Y. Let circle X be tangent to AD and CD at points H and G, respectively. Also, let circle Y be tangent to AB and BC at points E and F, respectively. Since the point of tangency on diagonal AC of circles X and Y is not known, we have only two congruences by Tangent Segments Theorem. That is, EB = BF and DG = DH.

We denote AE = a, EB = b, FC = c, CG = d, DG = e, and AH = f. Since ABCD is circumscribable, then by Theorem 3. 3, AB + CD = BC + AD. By substitution, a + b + d + e = b + c + e +f. Subtracting b + e from both sides of the equation, we get a + d = c + f (3) Since AC is tangent to the incircle of ? ABC, then by Tangent Segments Theorem, AC = a + c. (4) Similarly, AC is tangent to the incircle of ? ADC, thus by Tangent Segments Theorem, AC = d + f. (5) From (4) and (5), we have + c = d + f. (6) Subtracting corresponding sides of (3) from (6), we get c – d = d – c By combining similar terms, 2c = 2d Thus, c = d. Going back to the figure, we note that c = d means that the distances from C to the points of tangency of incircles X and Y on side AC are equal. This implies that the points of tangency of X and Y on AC are the same point. Therefore, we have shown that circle X is tangent to circle Y. Theorem 3. 8. The area of a circumscribable quadrilateral is equal to the product of its inradius and semiperimeter.

Figure 3. 12 Proof. Let ABCD be a circumscribable quadrilateral with inradius r (Figure 3. 12) and denote the area of quadrilateral ABCD by (ABCD). We want to show that [pic]. Let the incircle O intersect the sides AB, BC, CD, and DA at points E, F, G, and H, respectively. By the Tangent Segments Theorem, AE = AH, BE = BF, CF = CG and GD = DH. Let AE and AH be denoted by a, BE and BF by b, CF and CG by c and GD and DH by d. Let us denote the area of a triangle by its vertices enclosed in parentheses. For example, (ABC) means the area of triangle ABC.

Thus, using the formula for the area of a right triangle, [pic] [pic] [pic] [pic] The area of quadrilateral ABCD is equal to the sum of the areas of the eight triangles above, that is, [pic] By substitution, we get [pic]. Simplifying, [pic]. Then factor out r, [pic]. But [pic] is the semiperimeter of quadrilateral [pic] and [pic]. Therefore, by substitution, [pic]. Example 3. 9. A line cuts a circumscribable quadrilateral into two quadrilaterals with equal areas and equal perimeters. Prove that the line passes through the center of the inscribed circle. Figure 3. 13 Solution.

Let ABCD be a circumscribable quadrilateral with inscribed circle O. Consider a line that intersects AB and CD at points X and Y, respectively, such that AXYD and XBCY have equal areas and equal perimeters. (Figure 3. 13).

Hence, AX + XY + YD + DA = XB + BC + CY + XY. Subtracting XY from both sides of the equation, we get AX + YD + DA = XB + BC + CY. (7) Consider ? AOX. If r is the length of the inradius of circle O, then (AOX) = [pic] AX. Similarly, (YOD) = [pic]YD, (DOA) = [pic]DA, (XOB) = [pic]XB, (BOC) = [pic] BC, and (COY) = [pic]CY.

Multiplying both sides of (7) by [pic], we get [pic] AX + [pic]YD + [pic]DA =[pic] XB +[pic] BC + [pic]CY. By substitution, (AOX) + (YOD) + (DOA) = (XOB) + (BOC) + (COY) that is, (AXOYD) = (BXOYC). Because (AXOYD) + (BXOYC) = (ABCD) it follows that (AXOYD) = (BXOYC) = ? (ABCD). Suppose, on the contrary, that XY does not pass through the center of the inscribed circle and Figure 3. 14 assume that O lies in the interior of AXYD. (Figure 3. 15). We have, (BXYC) = ? (ABCD) = (BXOYC), But (BXOYC) = (BXYC) + (XOY), thus (XOY) = 0, which is a contradiction. CHAPTER 4 CONDITIONS FOR A QUADRILATERAL TO BE CIRCUMSCRIBABLE

In the preceding chapter, we have discussed the properties of circumscribable quadrilaterals and its proofs. This time, we look at some conditions that must be satisfied in order for a quadrilateral to be circumscribable. Theorem 4. 1. If the four angle bisectors of a quadrilateral are concurrent, then it is circumscribable. Figure 4. 1 Proof. Let ABCD be a quadrilateral with the four angle bisectors concurrent at point O (Figure 4. 1). We construct the perpendicular from O to the sides of ABCD and let the foot of the perpendiculars to the sides AB, BC, CD, and DA be points P, Q, R, and S, respectively.

Thus, OP[pic]AB, OQ[pic]BC, OR[pic]CD, and OS[pic]AD. Since O is on the angle bisector of [pic]A, [pic]B, [pic]C, and [pic]D, then by the Point on Angle Bisector Theorem, OS = OP, OP = OQ, OQ = OR, and OR = OS. By transitivity, OP = OQ = OR = OS. Thus, we can construct a circle passing through P, Q, R, and S whose center is at point O and whose radius is equal to the length of segment OP. Furthermore, since P is the foot of the perpendicular at AB, it implies that AB is tangent to circle O at P. Using the same reasoning, BC is tangent to O at Q, CD is tangent to O at R, and AD is tangent to O at S.

Thus we have shown that the four sides of ABCD are tangent to circle O. Therefore by definition, ABCD is circumscribable. Theorem 4. 2. If the sum of the lengths of a pair of opposite sides of a quadrilateral is equal to the sum of the lengths of the other pair of opposite sides, then the quadrilateral is circumscribable. Proof. Consider quadrilateral ABCD such that AB + CD = BC + AD. We must show that ABCD is circumscribable. To prove this, we consider two cases: AB = BC or AB [pic]BC. Case 1: If AB = BC, then since AB + CD = BC + AD, we get CD = AD.

This implies that ABCD must either be a kite or a rhombus and therefore circumscribable. (Figure 4. 2) Figure 4. 2 Case 2: Assume that AB [pic]BC and, without loss of generality, AB < BC. If AB < BC, then by substitution to the equation AB + CD = BC + AD, we can conclude that CD > AD. Since AB < BC, there exists a point P on BC such that AB = BP. Similarly, since CD > AD, there exists a point Q on CD such that AD = DQ (Figure 4. 3). Figure 4. 3 Since AB + CD = BC + AD, then by Segment Addition Postulate and substitution we have, AB + DQ + QC = BP + PC + AD.

But AB = BP and AD = DQ thus, QC = PC. With these results, we have formed three isosceles triangles on ABCD namely, ? ABP, ? PCQ, and ? ADQ (Figure 4. 4). Figure 4. 4 Next we construct the angle bisectors of [pic]B, [pic]C, and [pic]D. Since ? ABP, ? PCQ, and ? ADQ are isosceles then the angle bisectors of [pic]B, [pic]C, and [pic]D are also the perpendicular bisectors of the sides of ? APQ. But the perpendicular bisectors of the sides of a triangle are concurrent at the circumcenter. Therefore, we have shown that the three angle bisectors of ABCD are concurrent, in this case at point O. (Figure 4. 5). Figure 4. To show that the fourth angle bisector of the given quadrilateral is concurrent with the three other angle bisectors at O, we construct perpendicular segments from O to the sides of ABCD. Let the foot of the perpendiculars from O to the sides AB, BC, CD, and AD be denoted by H, I, J, and K, respectively (Figure 4. 6). Figure 4. 6 Since O is on the bisector of [pic]B, [pic]C, and [pic]D then, HO = IO, IO = OJ, and OJ = OK. By transitivity, HO = OJ but OJ = OK therefore we can say that HO = OK. Since HO and OK are perpendicular segments from O, then we can conclude that O lies on the bisector of [pic]A.

Thus, we have shown that the four angle bisectors of ABCD are concurrent at O. By Theorem 4. 1, ABCD is circumscribable. Theorem 4. 3. A quadrilateral is circumscribable if the incircles of the two triangles formed by a diagonal are tangent to each other. Figure 4. 7 Proof. Let ABCD be a quadrilateral with diagonal AC and the incircles of ? ADC and ? ABC be tangent at E (Figure 4. 7). We will show that AB + CD = AD + BC. Let the incircle of ? ABC intersect AB and BC at F and G, respectively. Further, let the incircle of ? ADC intersect

AD and DC at points H and I, respectively. We denote AB by a, FB by b, GC by c, and DI by d.

By the Tangent Segments Theorem, AE = AF = AH = a FB = BG = b GC = EC = IC = c and DI = DH = d. By substitution, AB = a + b BC = b + c CD = c + d AD = a + d. Thus, AB + DC = a + b + c + d and BC + AD = a + b + c + d which implies AB + DC = BC + AD. Therefore, by Theorem 4. 2, quadrilateral ABCD is circumscribable. CHAPTER 5 SUMMARY, CONCLUSION, AND RECOMMENDATION This paper presents the definition and properties of circumscribable quadrilaterals. It also enumerates the conditions that must be satisfied for a quadrilateral to be circumscribable.

These concepts build the foundation to learn the more complicated properties of this type of quadrilateral. Circumscribable quadrilaterals are quadrilaterals that have an inscribed circle. The properties of such quadrilaterals that have been proven and discussed in this paper are as follows: 1. a quadrilateral is circumscribable if and only if the sum of a pair of opposite sides is equal to the sum of the other pair of opposite sides; 2. the four angle bisectors of a quadrilateral are concurrent if and only if the quadrilateral is circumscribable; 3. quadrilateral is circumscribable if and only if the incircles of the two triangles formed by a diagonal are tangent to each other; and 4. the area of a circumscribable quadrilateral is equal to the product of its inradius and semiperimeter. It is recommended that researchers look into other properties of circumscribable quadrilaterals and how they could be applied in problem solving. Moreover, research could also be done on problems given in mathematical Olympiads about the said quadrilateral. ————————– P P P