Difference of squares of two natural numbers

One of the basic arithmetic operations is finding squares and difference between squares of two natural numbers. Though there are various methods to find the difference between squares of two natural numbers, still there are scopes to find simplified and easy approaches. As the sequence formed using the difference between squares of two natural numbers follow a number patterns, using number patterns may facilitate more easy approach. Also, this sequence has some general properties which are already discussed by many mathematicians in different notations. Apart from these, the sequence has some special properties like sequence – difference property, difference – sum property, which helps to find the value easily. The sequence also has some relations that assist to form a number pattern.

This paper tries to identify the general properties, special properties of finding difference between the squares of any two natural numbers using number patterns. A rhombus rule relationship between the sequences of numbers formed by considering the difference between squares of the two natural numbers has been defined. A new method to find a2 – b2 also has been introduced in some simple cases. This approach will help the secondary education lower grade students in identifying and recognizing number patterns and squares of natural numbers.

Mathematical Subject Classifications: (2010) 11A25, 11A51, 40C99, 03F50

DIFFERENCE BETWEEN SQUARES OF TWO NATURAL NUMBERS –

RELATIONS, PROPERTIES AND NEW APPROACH

Introduction

Mathematics, a subject of problem solving skills and applications, has wide usage in all the fields. Basic skills of mathematical applications in number systems used even in day – to – day life. Though calculators and computers have greater influences in calculations, still there is a need to find new easy methods of calculations to improve personal intellectual skills.

As there has been growing interest, in mathematics education, in teaching and learning, many mathematicians build simple and different methods, rules and relationships in various mathematical field. Though various investigations have made important contributions to mathematics development and education (2), there still room for new research to clarify the mutual relationship between the numbers and number patterns.

In natural numbers, various subsets have been recognized by ancient mathematicians. Some are odd numbers, prime numbers, oblong numbers, triangular numbers and squares. These numbers shall be identified by number patterns. Recognizing number patterns is also an important problem-solving skill. Working with number patterns leads directly to the concept of functions in mathematics: a formal description of the relationships among different quantities.

One of the basic arithmetic operations is finding squares and difference between squares of two natural numbers. Already many proofs and relationships were identified and proved in finding difference between squares of two natural numbers. We use different methods to find the difference between squares of two natural numbers. That is, to find a2 – b2. Though, this area of research may be discussed by early mathematicians and researchers in various aspects, still there are many interesting ways to discuss the same in teaching.

Teaching number patterns in secondary level education is most important issue as the students develop their analytical and cognitive skills in this stage. Different arithmetic operations and calculations need to be introduced in such way that they help the students in lifelong learning. Easy and simplified approaches will support the students in logical reasoning.

This paper tries to identify the general properties, special properties of finding difference between the squares of any two natural numbers using number patterns. Also, this paper tries to define the rhombus rule relationship between the sequences of numbers formed by the differences of squares of two natural numbers. A new method to find a2 – b2 also has been introduced in some simple cases.

These may be introduced in secondary school early grades, before introducing algebraic techniques of finding a2 – b2 to develop the knowledge and understanding of number patterns. This will help to recognize and apply number patterns in further level.

Literature Review

To find the difference between the squares of any two natural numbers, we use different methods. Also, we use various rules to find the square of a natural number. Some properties were also been identified by the researchers and mathematicians.

Methods used to find the difference between squares of two natural numbers

Direct Method

The difference between the squares of two natural numbers shall be found out by finding the squares of the numbers directly.

Example: 252 – 52 = 625 – 25 = 600

Using algebraic rule

The algebraic rule a2 – b2 = (a – b)(a + b) shall be applied to find the difference between the squares of two natural numbers.

Example: 252 – 52 = (25 – 5)(25 + 5) = 20 x 30 = 600

Method when a – b = 1(2)

“ The difference between the squares of every two consecutive natural numbers is always an odd number, and that it is equal to the sum of these numbers.”

Example: 252 – 242 = 25 + 24 = 49

Methods used to find the square of a natural number

Using Algebraic Method

The algebraic rules shall be used to find the square of natural number other than the direct multiplication. In general, (a + b)2, (a – b)2 are used to find the squares of a natural number from nearest whole number.

Example: 992 = (100 – 1)2

= 1002 – 2(100)(1) + 12 = 10000 – 200 + 1

= 9801

Square of a number using previous number(8)

The following rule may be applied to find the square of a number using previous number.

(n + 1)2 = n2 + n + (n+1)

Example: 312 = 302 + 30 + 31 = 900 + 30 + 31 = 961

The Gilbreth Method of finding square(9)

The Gilbreth method uses binomial theorem to find the square of a natural number. The rule is

n2 = 100(n – 25) + (50 – n)2

Example: 992 = 100(99 – 25) + (50 – 99)2

= 7400 + 2401 = 9801

Other than the above mentioned methods various methods are used based on the knowledge and requirements.

Properties of differences between squares of the natural numbers

2. 3. 1. The difference between squares of any two consecutive natural numbers is always odd.

To prove this property, let us consider two consecutive natural numbers, say 25 and 26

Now let us find 262 – 252

262 – 252 = (26 + 25)(26 – 25) [Using algebraic rule]

= 51 x 1 = 51, an odd number

2. 3. 2. The difference between squares of any two alternative natural numbers is always even.

To prove this property, let us consider two alternative natural numbers, say 125 and 127

Now let us find 1272 – 1252

1272 – 1252 = (127 + 125)(127 – 125) [Using algebraic rule]

= 252 x 2 = 504, an even number

Some other properties were also identified and discussed by various mathematicians and researchers.

Number Patterns and Difference Between the Squares of Two Natural Numbers – Discussions and Findings

Some of the properties stated above shall be proved by using number pattern. Number patterns are interesting area of arithmetic that stimulates the logical reasoning. They shall be applied in various notations to identify the sequences and relations between the numbers.

3. 1. Sample Table for the difference between squares of two natural numbers

To find the properties and relations that are satisfied by the sequences formed by the differences between the squares of two natural numbers, let us form a number pattern. For discussion purposes, let us consider first 10 natural numbers 1, 2, 3 … 10.

Now, let us find the difference between two consecutive natural numbers.

That is, 22 – 12 = 3; 32 – 22 = 5; and so on.

Then the sequence will be as follows: 3, 5, 7, 9, 11, 13, 15, 17 and 19.

The sequence is a set of odd numbers starting from 3.

i. e., Difference 1: {x| x is an odd number greater than or equal to 3, x Î N}

In the same way, let us form the sequence for the difference between squares of two alternative natural numbers.

That is, 32 – 12 = 8, 42 – 22 = 12, and so on.

Then the sequence will be: 8, 12, 16, 20, 24, 28, 32 and 36

Thus the sequence is a set of even numbers and multiples of 4 starting from 8.

i. e., Difference 2: {x| x is an multiple of 4 greater than or equal to 8, x Î N}

By proceeding this way, the sequences for other differences shall be formed.

Let us represent the sequences in a table for discussion purposes.

In Table 1, N is the natural number.

S is the square of the corresponding natural number.

D1 represents the difference between the squares of two consecutive natural numbers. That is, the difference between the numbers is 1.

D2 represents the difference between the squares of two alternate natural numbers. That is, the difference between the numbers is 2.

D3 represents the difference between the squares of 4th and 1st number. That is, the difference between the numbers is 3, and so on.

3. 2. Relationship between the row elements of each column

Now, let us discuss the relationship between the elements of rows and columns of the table.

From the above table,

Column D1 shows that the difference between squares of two consecutive numbers is odd.

Column D2 shows that the difference between squares of two alternate numbers is even.

The other columns show that the difference between the squares of two numbers is either odd or even.

From the above findings, the following properties shall be defined for the difference between squares of any two natural numbers.

3. 3. General Properties of the difference between squares of two natural numbers:

The difference between squares of any two consecutive natural numbers is always odd.

Proof: Column D1 proves this property.

This may also be tested randomly for big numbers.

Let us consider two digit consecutive natural numbers, say 96 and 97.

Now, 972 – 962 = 9409 – 9216

= 493, an odd number

Let us consider three digit consecutive natural numbers, say 757 and 758.

Thus, 7582 – 7572 = 574564 – 573049

= 1515, an odd number

This property may also be further tested for big numbers and proved. For example, let us consider five digit two consecutive natural numbers, say 15887 and 15888.

Then, 158882 – 158872 = 252428544 – 252396769

= 31775, an odd number

Apart from these, the property shall also be easily derived by the natural numbers properties. As the difference between two consecutive numbers is 1, the natural number property " The sum of odd and even natural numbers is always odd", shall be applied to prove this property.

The difference between squares of any two alternative natural numbers is always even.

Proof: Column D2 proves this property.

This may also be verified for big numbers by considering different digit natural numbers as discussed above.

Apart from this, as the difference between two alternate natural numbers is 2, the natural numbers property " A natural number said to be even if it is a multiple of two" shall also be used for proving the stated property.

The difference between squares of any two natural numbers is either odd or even, depending upon the difference between the numbers.

Proof: The other columns of Table 1 prove this property.

In Table 1, as D3 represents the sequence formed by the difference between two natural numbers whose difference is 3, an odd number, the sequence is also odd. Thus, the property may be proved by testing the other Columns D4, D5, …

Also, the addition, subtraction and multiplication properties of natural numbers prove this property.

Example:

112 – 62

Here the difference (11 – 6 = 5) is odd.

So, the result will be odd.

i. e. 112 – 62 = 121 – 36 = 85, an odd number

122 – 82

Here the difference (12 – 8 = 4) is even.

So, the result will be even.

i. e. 122 – 82 = 144 – 64 = 80, an even number

3. 4. Special Properties of the difference between squares of the two natural numbers

Table 1 also facilitates to find some special properties stated below.

Sequence Difference Property

Table 1 shows that the sequences formed are following a number pattern with a common property between them. Let us consider the number sequences of each column.

Let us consider the first column D1 elements. D1: 3, 5, 7, 9, 11 … …

As D1 represents the difference between the squares of two consecutive natural numbers, let us say, a and b with a > b, the difference between them will be 1.

That is a – b = 1

Let us consider the difference between the elements in the sequence.

The difference between the numbers in the sequence is 2.

Thus the difference between the elements of the sequence shall be expressed as, 2 x 1. Thus, Difference = 2(a – b)

Now, let us consider the second column D2 elements. D2: 8, 12, 16, 26, … … …

As D2 represents the difference between the squares of two alternative natural numbers, the difference between the natural numbers, say a and b is always 2. That is a – b = 2

If we consider the difference between the elements in the sequence, the difference is 4.

Thus, the difference between the elements in the sequence shall be expressed as 2 x 2.

That is, difference = 2 (a – b)

In the same way, D3: 15, 21, 27, 33, … … …

D3 represents the difference between squares of the 4th and 1st numbers, difference is 3. That is a – b = 3

The difference between the numbers in the sequence is 6.

Thus, difference = 2 x 3 = 2(a – b)

All other columns also show that the difference between the numbers in the corresponding sequence is 2 (a – b)

Thus, this may be generalized as following property:

“ The difference between elements of the number sequence, formed by the difference between any two natural numbers, is equal to two times of the difference between those corresponding natural numbers.”

Difference – Sum Property:

From Table 1, we shall also identify another relationship between the elements of the sequence formed.

Let us consider the columns from table 1 other than D1.

Consider D2: 8, 12, 16, 20, …

This sequence shall be formed by adding two numbers of Column D1.

i. e. 8 = 3 + 5

12 = 5 + 7

16 = 7 + 9

20 = 9 + 11

And so on.

Thus, if the difference between the natural numbers taken is 2, then the number sequence of the difference between the two natural numbers shall be formed by adding 2 natural numbers.

Consider D3: 15, 21, 27, …

This sequence shall be formed by adding three numbers from Column D1.

i. e. 15 = 3 + 5 + 7

21 = 5 + 7 + 9

27 = 7 + 9 + 11

And so on.

Thus, if the difference between the natural numbers taken is 3, then the number sequence of the difference between the two natural numbers shall be formed by adding 3 natural numbers.

This may also be verified with respect to the other columns.

Table 2 shows the above relationship between the differences of the squares of the natural numbers.

Now the above relation shall be generalized as

“ If a – b = k > 1, then a2 – b2 shall be written as the sum of ‘ k’ natural numbers”

As Column D1 elements are odd natural numbers, this property may be defined as

“ If a – b = k > 1, then a2 – b2 shall be written as the sum of ‘ k’ odd natural numbers”

As these odd numbers are consecutive, the property may be further precisely defined as:

“ If a – b = k > 1, then a2 – b2 shall be written as the sum of ‘ k’ consecutive odd natural numbers”

3. 5. New Method to find the difference between squares of two natural numbers

Using the above difference – sum property, the difference between squares of two natural numbers shall be found as follows.

The property shows that, a2 – b2 is equal to sum of ‘ k’ consecutive odd numbers. Now, the principal idea is to find those ‘ k’ consecutive odd numbers.

Let us consider two natural numbers, say 7 and 10.

The difference between them 10 – 7 = 3

Thus, 102 – 72 = sum of three consecutive odd numbers.

102 – 72 = 100 – 49 = 51

Now, 51 = Sum of 3 consecutive odd numbers

i. e., 51 = 15 + 17 + 19

Let we try to find these 3 numbers with respect to either the first number, let us say, ‘ a’ or the second number, say, ‘ b’.

Assume, for ‘ b’

As general form for odd numbers is either (2n + 1) or (2n – 1), as b < a, consider (2n + 1) form.

15 = 2(7) + 1 = 2b + 1

17 = 2(7) + 3 = 2b + 3

19 = 2(7) + 5 = 2b + 5

Thus, 102 – 72 shall be written as the sum of 3 consecutive odd numbers starting from 15.

i. e. starting from 2b + 1

This idea may also be applied for higher digit numbers. Let us consider two 3 digit numbers, 101 and 105. Let us find 1052 – 1012

Here the difference is 4. Thus 1052 – 1012 shall be written as the sum of 4 consecutive odd numbers.

The numbers shall be found as follows:

Here b = 101

The first odd number = 2b + 1 = 2(101) + 1 = 203

Thus, the 4 consecutive odd numbers are: 203, 205, 207, 209

So,

1052 – 1012 = 203 + 205 + 207 + 209 = 824

This shall be verified for any number of digits. Let us consider two 6 digit numbers 100519, 100521. Let us find 1005212 – 1005192

Here the difference is 2. Thus 1005212 – 1005192 shall be written as the sum of two odd numbers.

Applying the same idea,

The first odd number = 2(100519) + 1 = 201039

Thus the 2 consecutive odd numbers are: 201039, 201041

1005212 – 1005192 = 201039 + 201041 = 402080

The above result shall be verified by using other methods.

For example: 1052 – 1012

1052 – 1012 = 11025 – 10201 = 824 (Using Direct Method)

1052 – 1012 = (105 + 101) (105 – 101) = 206 x 4 = 824 (Using Algebraic Rule)

Thus, this idea shall be generalized as follows:

“ a2 – b2shall be found by adding the (a – b) consecutive odd numbers starting from 2b + 1”

This shall also be found using the first term ‘ a’. As a > b, let us consider (2n – 1) form of odd numbers.

From Table 1, 102 – 62 = 13 + 15 + 17 + 19 = 64

Here, 2a – 1 = 2(10) – 1 = 19

2a – 3 = 2(10) – 3 = 17

2a – 5 = 2(10) – 5 = 15

2a – 7 = 2(10) – 7 = 13

Thus, as the difference between the numbers is 4, 102 – 62 shall be written as the sum of four consecutive odd numbers in reverse order starting from 2a – 1.

Thus proceeding, this may be generalized as,

“ a2 – b2shall be found by adding the (a – b) consecutive odd numbers starting from 2a – 1 in reverse order”

Finding the first number of each column

Let us check the number pattern followed by the first numbers of each column. From Table 1, the first numbers of each column are: 3, 8, 15, 24 …

Let us find the difference between elements of this sequence.

The difference between two consecutive terms of this sequence is 5, 7, 9 …

i. e. D2 – D1 = 8 – 3 = 5; D3 – D2 = 7; D4 – D3 = 9 and so on.

As D2 represents the difference between two alternate natural numbers, (say a and b) which implies that the difference between a and b is 2.

Now, 5 = 2 (2) +1

i. e. 2 times of the difference between the numbers + 1

In the same idea, D3 – D2 = 15 – 8 = 7

As D3 represents the difference between squares of the 4th and 1st natural numbers, (say a and b) which implies that the difference between a and b is 3.

Thus, 7 = 2(3) + 1

This also shows that the difference shall be found by

= 2 times of the difference between the numbers + 1

Thus,

" The first term of the each column shall be found by adding the previous column first term with 2 times of the difference between the numbers + 1"

Finding the elements row – wise

The elements of the table shall also be formed in row wise.

If we check the elements of each row, we can find that they follow a number pattern sequence with some property.

Let us consider the elements of row when N = 5: 20, 40, 60, 80

20 = 2 x 5 x 2

Here, 5 represent the row natural number.

2 represent the difference between the elements using which the column is formed.

Thus Row element = 2 x N x difference

In the same way, 40 = 2 x 5 x 4

= 2 x N x difference

Thus, the elements shall be formed by the rule:

“ Row Element = 2 x N x difference”

This shall be applied for middle rows also.

For example, let us consider the row between 5 & 6:

The elements in this intermediate row are: 11, 33, 55, 77, 99

Here N is the mid value of 5 & 6. i. e. N = 5. 5

Let us consider the elements and apply the above stated rule.

11 = 2 x N x difference

= 2 x 5. 5 x 1

In the same way other elements shall also be formed.

Thus the elements of the table shall be formed in row wise using the stated rule.

Rhombus Rule Relation

Let us consider the elements in D2, D3 and D4.

Consider the elements in the rhombus drawn, 24, 33, 39 and 48

24 + 48 = 72

33 + 39 = 72

Thus the sums of the elements in the opposite corners are equal.

The other column elements also prove the same.

Thus, Rhombus Rule Relation:

“ Sum of the elements the same row of the sequence of alternative columns is equal to the sum of the two elements in the intermediate column”

Application of the Properties in Finding the Square of a number

The square of a natural number shall be found by various methods. Here is one of the suggested methods.

This method uses nearest 10’s and 100’s to find the square of a number.

This method is also based on the algebraic formula a2 – b2 = (a – b)(a + b)

If a > b, b2 = a2 – (a2 – b2)

If b > a, b2 = a2 + (b2 – a2)

Example: Square of 32

As we need to find 322, let us assume b = 32.

The nearest multiple of 10 is 30. Let a = 30

Here b > a. b2 = a2 + (b2 – a2)

322 = 302 + (322 – 302)

Using the Difference – Sum Property,

322 = 900 + 61 + 63 = 1024

Example 2: Square of 9972

Let b = 997

Nearest multiple 10 is 1000. Let a = 1000

Here a > b, so b2 = a2 – (a2 – b2)

9972 = 10002 – (10002 – 9972)

Using Difference – Sum Property,

9972 = 1000000 – (1995 + 1997 + 1999)

= 994009

Conclusion

Though this method shall be applied to find the difference between squares of any two natural numbers, if the difference is big, it will be cumbersome. Thus, this method shall be used for finding the difference between squares of any two natural numbers where the difference is manageable. The properties shall be used for easy calculation.

This properties and approach shall be introduced in secondary school lower grade levels, to make the students to identify the number patterns. This approach will surely help the students to understand the properties of squares, difference and natural numbers. The new approach will surely help the students in developing their reasoning skills.

Limitations

As number systems, number patterns and arithmetic operations have wide applications in various fields, the above properties, rules and relations shall be further studied intensively based on the requirements. Thus, new properties and relations shall be identified and discussed with respect to other nations.