Maxima and minima of functions

This term paper presents concise explanations of the subject's general principles and uses worked examples freely to expand the ideas about solving the problems by suitable methods. Each example shows the method of obtaining the solution and includes additional explanatory techniques. For some topics, where it would have been difficult to understand a solution given on a single problem, the solution has been drawn in step-by-step form. All the figures used have been taken from Google Book search.

The term paper covers the necessary definitions on MAXIMA AND MINIMA OF THE FUNCTIONS and some of its important applications. It covers the topic such as types of other method for solving the big problem in a shortcut method known . The aspects of how to develop some of the most commonly seen problems is also covered in this term paper. The motive of this term paper is make the reader familiar with the concepts of application of maxima and minima of the function and where this is used. Focus has been more on taking the simpler problem so that the concept could be made clearer even to the beginners to engineering mathematics.

MAXIMA AND MINIMA

The diagram below shows part of a function y = f(x).

The point A is a local maximum and the point B is a local minimum.

At each of these points the tangent to the curve is parallel to the

X – axis so the derivative of the function is zero. Both of these points

are therefore stationary points of the function. The term local is used since these points are the maximum and minimum in this particular

Region.

The rate of change of a function is measured by its derivative.

When the derivative is positive, the function is increasing,

When the derivative is negative, the function is decreasing.

Thus the rate of change of the gradient is measured by its derivative,

Which is the second derivative of the original function?

Functions can have " hills and valleys": places where they reach a minimum or maximum value.

It may not be the minimum or maximum for the whole function, but locally it is.

You can see where they are,

but how do we define them?

Local Maximum

First we need to choose an interval:

Then we can say that a local maximum is the point where:

The height of the function at " a" is greater than (or equal to) the height anywhere else in that interval.

Or, more briefly:

f(a) â‰¥ f(x) for all x in the interval

In other words, there is no height greater than f(a).

Note: f(a) should be inside the interval, not at one end or the other.

Local Minimum

Likewise, a local minimum is:

f(a) â‰¤ f(x) for all x in the interval

The plural of Maximum is Maxima

The plural of Minimum is Minima

Maxima and Minima are collectively called Extrema

Global (or Absolute) Maximum and Minimum

The maximum or minimum over the entire function is called an " Absolute" or " Global" maximum or minimum.

There is only one global maximum (and one global minimum) but there can be more than one local maximum or minimum.

Assuming this function continues downwards to left and right:

The Global Maximum is about 3. 7

The Global Minimum is -Infinity

Maxima and Minima of Functions of Two Variables

Locate relative maxima, minima and saddle points of functions of two variables. Several examples with detailed solutions are presented. 3-Dimensional graphs of functions are shown to confirm the existence of these points. More on Optimization Problems with Functions of Two Variables in this web site.

Theorem

Let f be a function with two variables with continuous second order partial derivativesfxx, fyyand fxyat a critical point (a, b). Let

D = fxx(a, b) fyy(a, b) – fxy2(a, b)

If D > 0 and fxx(a, b) > 0, then f has a relative minimum at (a, b).

If D > 0 and fxx(a, b) <0, then f has a relative maximum at (a, b).

If D <0, then f has a saddle point at (a, b).

If D = 0, then no conclusion can be drawn.

We now present several examples with detailed solutions on how to locate relative minima, maxima and saddle points of functions of two variables. When too many critical points are found, the use of a table is very convenient.

Example 1: Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by

f(x , y) = 2×2+ 2xy + 2y2- 6x

Solution to Example 1:

Find the first partial derivatives fxand fy.

fx(x, y) = 4x + 2y – 6

fy(x, y) = 2x + 4y

The critical points satisfy the equations fx(x, y) = 0 and fy(x, y) = 0 simultaneously. Hence.

4x + 2y – 6 = 0

2x + 4y = 0

The above system of equations has one solution at the point (2,-1).

We now need to find the second order partial derivatives fxx(x, y), fyy(x, y) and fxy(x, y).

fxx(x, y) = 4

fxx(x, y) = 4

fxy(x, y) = 2

We now need to find D defined above.

D = fxx(2,-1) fyy(2,-1) – fxy2(2,-1) = (4)(4) – 22= 12

Since D is positive and fxx(2,-1) is also positive, according to the above theorem function f has a local minimum at (2,-1).

The 3-Dimensional graph of function f given above shows that f has a local minimum at the point (2,-1, f(2,-1)) = (2,-1,-6).

Example 2: Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by

f(x , y) = 2×2- 4xy + y4+ 2

Solution to Example 2:

Find the first partial derivatives fxand fy.

fx(x, y) = 4x – 4y

fy(x, y) = – 4x + 4y3

Determine the critical points by solving the equations fx(x, y) = 0 and fy(x, y) = 0 simultaneously. Hence.

4x – 4y = 0

– 4x + 4y3= 0

The first equation gives x = y. Substitute x by y in the equation – 4x + 4y3= 0 to obtain.

– 4y + 4y3= 0

Factor and solve for y.

4y(-1 + y2) = 0

y = 0 , y = 1 and y = -1

We now use the equation x = y to find the critical points.

(0 , 0) , (1 , 1) and (-1 , -1)

We now determine the second order partial derivatives.

fxx(x, y) = 4

fyy(x, y) = 12y2

fxy(x, y) = -4

We now use a table to study the signs of D and fxx(a, b) and use the above theorem to decide on whether a given critical point is a saddle point, relative maximum or minimum.

critical point (a, b)

(0, 0)

(1, 1)

(-1, 1)

fxx(a, b)

4

4

4

fyy(a, b)

0

12

12

fxy(a, b)

-4

-4

-4

D

-16

32

32

saddle point

relative minimum

relative minimum

A 3-Dimensional graph of function f shows that f has two local minima at (-1,-1, 1) and (1, 1, 1) and one saddle point at (0, 0, 2).

Example 3: Determine the critical points and locate any relative minima, maxima and saddle points of function f defined by f(x , y) = – x4- y4+ 4xy

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Solution to Example 3:

First partial derivatives fxand fyare given by.

fx(x, y) = – 4×3+ 4y

fy(x, y) = – 4y3+ 4x

We now solve the equations fy(x, y) = 0 and fx(x, y) = 0 to find the critical points..

– 4×3+ 4y = 0

– 4y3+ 4x = 0

The first equation gives y = x3. Combined with the second equation, we obtain.

– 4(x3)3+ 4x = 0

Which may be written as .

x(x4- 1)(x4+ 1) = 0

Which has the solutions.

x = 0 , -1 and 1.

We now use the equation y = x3to find the critical points.

(0 , 0) , (1 , 1) and (-1 , -1)

We now determine the second order partial derivatives.

fxx(x, y) = -12×2

The First Derivative: Maxima and Minima

Consider the function f(x)= 3×4âˆ’4×3âˆ’12×2+3 on the interval [âˆ’23]. We cannot find regions of which f is increasing or decreasing, relative maxima or minima, or the absolute maximum or minimum value of f on [âˆ’23] by inspection. Graphing by hand is tedious and imprecise. Even the use of a graphing program will only give us an approximation for the locations and values of maxima and minima. We can use the first derivative of f, however, to find all these things quickly and easily.

Increasing or Decreasing?

Let f be continuous on an interval I and differentiable on the interior of I.

If f(x)0 for all xI, then f is increasing on I.

If f(x)0 for all xI, then f is decreasing on I.

Example

The function f(x)= 3×4âˆ’4×3âˆ’12×2+3 has first derivative f(x) = = = 12×3âˆ’12×2âˆ’24x 12x(x2âˆ’xâˆ’2) 12x(x+1)(xâˆ’2)   Thus, f(x) is increasing on (âˆ’10)(2) and decreasing on (âˆ’âˆ’1)(02).

Relative Maxima and Minima

Relative extrema of f occur at critical points of f, values x0 for which either f(x0)= 0 or f(x0) is undefined.

First Derivative Test

Suppose f is continuous at a critical point x0.

If f(x)0 on an open interval extending left from x0 and f(x)0 on an open interval extending right from x0, then f has a relative maximum at x0.

If f(x)0 on an open interval extending left from x0 and f(x)0 on an open interval extending right from x0, then f has a relative minimum at x0.

If f(x) has the same sign on both an open interval extending left from x0 and an open interval extending right from x0, then f does not have a relative extremum at x0.

In summary, relative extrema occur where f(x) changes sign.

Example

Our function f(x)= 3×4âˆ’4×3âˆ’12×2+3 is differentiable everywhere on [âˆ’23], with f(x)= 0 for x= âˆ’102. These are the three critical points of f on [âˆ’23]. By the First Derivative Test, f has a relative maximum at x= 0 and relative minima at x= âˆ’1 and x= 2.

Absolute Maxima and Minima

If f has an extreme value on an open interval, then the extreme value occurs at a critical point of f.

If f has an extreme value on a closed interval, then the extreme value occurs either at a critical point or at an endpoint.

According to the Extreme Value Theorem, if a function is continuous on a closed interval, then it achieves both an absolute maximum and an absolute minimum on the interval.

Example

Since f(x)= 3×4âˆ’4×3âˆ’12×2+3 is continuous on [âˆ’23], f must have an absolute maximum and an absolute minimum on [âˆ’23]. We simply need to check the value of f at the critical points x= âˆ’102 and at the endpoints x= âˆ’2 and x= 3: f(âˆ’2) f(âˆ’1) f(0) f(2) f(3) = = = = = 35 âˆ’2 3 âˆ’29 30 Thus, on [âˆ’23], f(x) achieves a maximum value of 35 at x= âˆ’2 and a minimum value of -29 at x= 2.

We have discovered a lot about the shape of f(x)= 3×4âˆ’4×3âˆ’12×2+3 without ever graphing it! Now take a look at the graph and verify each of our conclusions.

APPLICATION AND CONCLUSION

The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Maximum means upper bound or largest possible quantity. The absolute maximum of a function is the largest number contained in the range of the function. That is, if f(a) is greater than or equal to f(x), for all x in the domain of the function, then f(a) is the absolute maximum. For example, the function f(x) = -16×2 + 32x + 6 has a maximum value of 22 occurring at x = 1. Every value of x produces a value of the function that is less than or equal to 22, hence, 22 is an absolute maximum. In terms of its graph, the absolute maximum of a function is the value of the function that corresponds to the highest point on the graph. Conversely, minimum means lower bound or least possible quantity. The absolute minimum of a function is the smallest number in its range and corresponds to the value of the function at the lowest point of its graph. If f(a) is less than or equal to f(x), for all x in the domain of the function, then f(a) is an absolute minimum. As an example, f(x) = 32×2 – 32x – 6 has an absolute minimum of -22, because every value of x produces a value greater than or equal to -22.

In some cases, a function will have no absolute maximum or minimum. For instance the function f(x) = 1/x has no absolute maximum value, nor does f(x) = -1/x have an absolute minimum. In still other cases, functions may have relative (or local) maxima and minima. Relative means relative to local or nearby values of the function. The terms relative maxima and relative minima refer to the largest, or least, value that a function takes on over some small portion or interval of its domain. Thus, if f(b) is greater than or equal to f(b ± h) for small values of h, then f(b) is a local maximum; if f(b) is less than or equal to f(b ± h), then f(b) is a relative minimum. For example, the function f(x) = x4 -12×3 – 58×2 + 180x + 225 has two relative minima (points A and C), one of which is also the absolute minimum (point C) of the function. It also has a relative maximum (point B), but no absolute maximum.

Finding maxima or minima also has important applications in linear algebra and game theory. For example, linear programming consists of maximizing (or minimizing) a particular quantity while requiring that certain constraints be imposed on other quantities. The quantity to be maximized (or minimized), as well as each of the constraints, is represented by an equation or inequality. The resulting system of equations or inequalities, usually linear, often contains hundreds or thousands of variables. The idea is to find the maximum value of a particular variable that represents a solution to the whole system. A practical example might be minimizing the cost of producing an automobile given certain known constraints on the cost of each part, and the time spent by each laborer, all of which may be interdependent. Regardless of the application, though, the key step in any maxima or minima problem is expressing the problem in mathematical terms.