Usefulness of maxima and minima of functions engineering essay

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The mathematical concept of a function expresses the intuitive idea that one quantity(input) completely determines another quantity (output). A function assigns a unique value or output to each input of a specified type. The argument and the value may be real numbers, but they can also be elements from any given sets: the domain and the co-domain of the function.

Whenever a relationship exists between two variables (or quantities) such that for every value of the first, there is only one corresponding value of the second, then we say: The second variable is a function of the first variable.

The first variable is the independent variable (usually x), and the second variable is the dependent variable (usually y).

The independent variable and the dependent variable are real numbers.

The term function is just a type of operator which transforms the given input to output according to the given conditions. This operator relates two or more quantities to each other, the quantities are known as variables. Out of total variables only one is independent and all other are dependent on that variable. One precise definition of a function is that it consists of an ordered triple of sets, which may be written as (X, Y, F). X is the domain of the function, Y is the co-domain, and F is a set of ordered pairs. In each of these ordered pairs (a, b), the first element 'a' is from the domain, the second element 'b' is from the co-domain, and every element in the domain is the first element in one and only one ordered pair. The set of all 'b' is known as the image of the function. Some authors use the term " range" to mean the image, others to mean the co-domain.

The notation Æ’: Xâ†’Y indicates that Æ’ is a function with domain X and co-domain Y.

(Domain implies input whereas range or co-domain implies output.)

In most practical situations, the domain and co-domain are understood from context, and only the relationship between the input and output is given. Thus

is usually written as

Here the two variables are 'x' and 'y' out of which 'x' is independent and 'y' is dependent on 'x'. From the other side if we consider 'y' as independent variable then 'x' is dependent on 'y'.

Every function can be plotted on graph or more precisely Argand Plain. The graph of function may be a straight line, a continuous curve, a circle, an ellipse or even a point also.

HISTORY OF MAXIMA AND MINIMA:

Since origin of life, all people knew, talked, applied the concept of maxima and minima in their daily lives without even knowing about the concept of maxima and minima. In the earlier phase of time the kings used to estimate the maximum and minimum army of the opposite side, doctors used to record minimum and maximum symptom of any disease, cooks used to estimate the maximum and minimum quantity of food or people before any function, the businessmen used to estimate maximum and minimum profit or loss in any transaction. Even today also the women in the house prepare the food according to maximum or minimum consumption by each individual.

Sir Issac Newton, a great scientist, invented the concept of functions and hence concept of maxima or minima. Since then his concepts are very usefully applicable in our daily lives.

PRESENT TIME CONCEPTS OF MAXIMA AND MINIMA:

The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Maximum means upper bound or largest possible quantity. The absolute maximum of a function is the largest number contained in the range of the function. That is, if f(a) is greater than or equal to f(x), for all x in the domain of the function, then f(a) is the absolute maximum. For example, the function f(x) = -16x2 + 32x + 6 has a maximum value of 22 occurring at x = 1. Every value of x produces a value of the function that is less than or equal to 22, hence, 22 is an absolute maximum. In terms of its graph, the absolute maximum of a function is the value of the function that corresponds to the highest point on the graph. Conversely, minimum means lower bound or least possible quantity. The absolute minimum of a function is the smallest number in its range and corresponds to the value of the function at the lowest point of its graph. If f(a) is less than or equal to f(x), for all x in the domain of the function, then f(a) is an absolute minimum. As an example, f(x) = 32x2 - 32x - 6 has an absolute minimum of -22, because every value of x produces a value greater than or equal to -22.

In some cases, a function will have no absolute maximum or minimum. For instance the function f(x) = 1/x has no absolute maximum value, nor does f(x) = -1/x have an absolute minimum. In still other cases, functions may have relative (or local) maxima and minima. Relative means relative to local or nearby values of the function. The terms relative maxima and relative minima refer to the largest, or least, value that a function takes on over some small portion or interval of its domain. Thus, if f(b) is greater than or equal to f(b ± h) for small values of h, then f(b) is a local maximum; if f(b) is less than or equal to f(b ± h), then f(b) is a relative minimum.

Finding the maxima and minima, both absolute and relative, of various functions represents an important class of problems solvable by use of differential calculus. The theory behind finding maximum and minimum values of a function is based on the fact that the derivative of a function is equal to the slope of the tangent. When the values of a function increase as the value of the independent variable increases, the lines that are tangent to the graph of the function have positive slope, and the function is said to be increasing. Conversely, when the values of the function decrease with increasing values of the independent variable, the tangent lines have negative slope, and the function is said to be decreasing. Precisely at the point where the function changes from increasing to decreasing or from decreasing to increasing, the tangent line is horizontal (has slope 0), and the derivative is zero (With reference to figure 1, the function is decreasing to the left of point A, as well as between points B and C, and increasing between points A and B and to the right of point C). In order to find maximum and minimum points, first find the values of the independent variable for which the derivative of the function is zero, then substitute them in the original function to obtain the corresponding maximum or minimum values of the function. Second, inspect the behavior of the derivative to the left and right of each point.

A wide variety of problems can be solved by finding maximum or minimum values of functions. For example, suppose it is desired to maximize the area of a rectangle inscribed in a semicircle. The area of the rectangle is given by A = 2xy. The semicircle is given by x2 + y2 = r2, for y â‰¥ 0, where r is the radius. To simplify the mathematics, note that A and A2 are both maximum for the same values of x and y, which occurs when the corner of the rectangle intersects the semicircle, that is, when y2 = r2 - x2. Thus, we must find a maximum value of the function A2 = 4x2(r2 -x2) = 4r2x2 - 4x4. The required condition is that the derivative be equal to zero, that is, d(A2)/dx = 8r2x - 16x3 = 0. This occurs when x = 0 or when x = 1â„ 2(r âˆš +2). Clearly the area is a maximum when x = 1â„ 2(r âˆš +2). Substitution of this value into the equation of the semicircle gives y = 1â„ 2(r âˆš +2), that is, y = x. Thus, the maximum area of a rectangle inscribed in a semicircle is A = 2xy = r2.

The problem of determining the maximum or minimum of function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century.

Let us recall the procedure for the case of a function of one variable y= f(x). First, we determine points where f'(x)= 0. These points are called critical points. At critical points the tangent line is horizontal. This is shown in the figure below.

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The second derivative test is employed to determine if a critical point is a relative maximum or a relative minimum. If f''()> 0, then 'x' is a relative minimum. If f''()<0, then 'x' is a maximum. If f''()= 0, then the test gives no information.

The notions of critical points and the second derivative test carry over to functions of two variables. Let z= f(x, y). Critical points are points in the xy-plane where the tangent plane is horizontal.

Since the normal vector of the tangent plane at (x, y) is given by

The tangent plane is horizontal if its normal vector points in the z direction. Hence, critical points are solutions of the equations:

because horizontal planes have normal vector parallel to z-axis. The two equations above must be solved simultaneously.

The Second Derivative Test for Functions of Two Variables

How can we determine if the critical points found above are relative maxima or minima? We apply a second derivative test for functions of two variables.

Let (x, y) be a critical point and define

We have the following cases:

If D> 0 and (,).)<0, then f(x, y) has a relative maximum at (,).).

If D> 0 and (,).)> 0, then f(x, y) has a relative minimum at (,).).

If D <0, then f(x, y) has a saddle point at (,).

If D= 0, the second derivative test is inconclusive.

Maxima and Minima in a Bounded Region

Suppose that our goal is to find the global maximum and minimum of our model function above in the square -2 <= x <= 2 and -2 <= y <= 2? There are three types of points that can potentially be global maxima or minima:

Relative extrema in the interior of the square.

Relative extrema on the boundary of the square.

Corner Points.

We have already done step 1. There are extrema at (1, 0) and (-1, 0). The boundary of square consists of 4 parts. Side 1 is y=-2 and -2 <= x <= 2. On this side, we have

The original function of 2 variables is now a function of x only. We set g'(x)= 0 to determine relative extrema on Side 1. It can be shown that x= 1 and x=-1 are the relative extrema. Since y=-2, the relative extrema on Side 1 are at (1,-2) and (-1,-2).

On Side 2 (x=-2 and -2 <= y <= 2)

We set h'(y)= 0 to determine the relative extrema. It can be shown that y= 0 is the only critical point, corresponding to (-2, 0).

We play the same game to determine the relative extrema on the other 2 sides. It can be shown that they are (2, 0), (1, 2), and (-1, 2).

Finally, we must include the 4 corners (-2,-2), (-2, 2), (2,-2), and (2, 2). In summary, the candidates for global maximum and minimum are (-1, 0), (1, 0), (1,-2), (-1,-2), (-2, 0), (2, 0), (1, 2), (-1, 2), (-2,-2), (-2, 2), (2,-2), and (2, 2). We evaluate f(x, y) at each of these points to determine the global max and min in the square. The global maximum occurs (-2, 0) and (1, 0). This can be seen in the figure above. The global minimum occurs at 4 points: (-1, 2), (-1,-2), (2, 2), and (2,-2).

One of the great powers of calculus is in the determination of the maximum or minimum value of a function. Take f(x) to be a function of x. Then the value of x for which the derivative of f(x) with respect to x is equal to zero corresponds to a maximum, a minimum or an inflexion point of the function f(x).

The derivative of a function can be geometrically interpreted as the slope of the curve of the mathematical function y(t) plotted as a function of t. The derivative is positive when a function is increasing toward a maximum, zero (horizontal) at the maximum, and negative just after the maximum. The second derivative is the rate of change of the derivative, and it is negative for the process described above since the first derivative (slope) is always getting smaller. The second derivative is always negative for a " hump" in the function, corresponding to a maximum.

A critical point (x, y) of f is a point where both the partial derivatives of the functions vanish. A local maximum, or a local minimum, is a critical point. In one variable, local maxima and minima are the only `nondegenerate' critical points. In two or more variables, other possibilities appear. For instance one has the saddle point, like the critical point of at (0; 0). In some directions this looks like a maximum, in other directions this looks like a minimum. We try to classify critical points by looking at the second derivatives.

APPLICATIONS OF MAXIMA AND MINIMA IN DAILY LIFE:

There are numerous practical applications in which it is desired to find the maximum or minimum value of a particular quantity. Such applications exist in economics, business, and engineering. Many can be solved using the methods of differential calculus described above. For example, in any manufacturing business it is usually possible to express profit as a function of the number of units sold. Finding a maximum for this function represents a straightforward way of maximizing profits. In other cases, the shape of a container may be determined by minimizing the amount of material required to manufacture it. The design of piping systems is often based on minimizing pressure drop which in turn minimizes required pump sizes and reduces cost. The shapes of steel beams are based on maximizing strength.

Finding maxima or minima also has important applications in linear algebra and game theory. For example, linear programming consists of maximizing (or minimizing) a particular quantity while requiring that certain constraints be imposed on other quantities. The quantity to be maximized (or minimized), as well as each of the constraints, is represented by an equation or inequality. The resulting system of equations or inequalities, usually linear, often contains hundreds or thousands of variables. The idea is to find the maximum value of a particular variable that represents a solution to the whole system. A practical example might be minimizing the cost of producing an automobile given certain known constraints on the cost of each part, and the time spent by each laborer, all of which may be interdependent. Regardless of the application, though, the key step in any maxima or minima problem is expressing the problem in mathematical terms.

Everything in this world is based on the concept of maxima and minima, every time we always calculate the maximum and minimum of every data. Now-a-days results are also based on the concepts of grades which is again based on the concept of maxima and minima.