Example of arrows impossibility theorem essay

[Law](), [Evidence]()

Introduction

Arrow's theorem (also known as " Arrow's paradox") is a theorem about the impossibility of a " collective choice ". Formulated by the American economist Kenneth Arrow in 1951.

The meaning of this theorem is that in the ordinal approach, there is no method of combining individual preferences for three or more alternatives that would satisfy some quite fair conditions and would always give a logically consistent result.

Ordinal approach is based on the fact that the preferences of the individual with respect to the choice of the proposed alternatives can not be measured quantitatively, but only qualitatively, that is, one alternative is better or worse than the other.

As part of the cardinal approach that measurability of preferences, Arrow's theorem in general does not work.

Formulations

Formulation of 1951.

Suppose there are N ≥ 2 voters voting for n ≥ 3 candidates (in terms of decision theory, the candidates are called alternatives). Each voter has an ordered list of alternatives. Electoral system - a function that transforms a set of N such lists (profile ballot) to the total ordered list.

The system of elections may have the following properties:

Versatility

For each profile there is the result of a vote - an ordered list of n alternatives.

Completeness

The voting system can produce as a result of all n! permutations of the alternatives.

Monotony

If all N lists some alternative x will remain in place, or rise above, and the rest of the order will not change in the list of x should remain in place or climb.

The absence of a dictator

No voter preference which would determine the outcome of the election, regardless of the preferences of other voters.

The independence of irrelevant alternatives

If a profile of voting changes so that alternatives x and y in all N lists will remain in the same order, then do not change their order, and in the final result.

THEN

For $N \geq 2$ and $n \geq 3$ there is no voting system that meets all five conditions.

Formulation of 1963.

The statments in 1963 are as follows:

Versatility

The absence of a dictator

The independence of irrelevant alternatives

Pareto efficiency, or the principle of unanimity. If every voter in the list of alternative x is above y, the same should be in the final result.

THEN

For $N \geq 2$ and $n \geq 3$ there is no voting system that meets all four conditions.

Arrow’s Impossibility Theorem Proof

We introduce the following notation:

$\succ_i$ - preferences of i-th agent; $[\succ']$ - profile of preferences (tuple whose elements are the preferences of the agents);

$W: L^n \rightarrow L$ - social welfare function; $\succ_W$ - collective preferences.

We denote O - the set of outcomes that each agent ranks according to their preferences.

We give a formal definition:

Pareto efficiency

W Pareto efficient if for any outcomes $o_1, o_2 \in O$, $\forall i\ (o_1 \succ_i o_2) \Rightarrow (o_1 \succ_W o_2)$

The independence of irrelevant alternatives

W is independent of irrelevant alternatives, if for any outcomes $o_1, o_2 \in O$, and for any two profiles, preferences $[\succ']$ and $[\succ''] \in L^n$, $\forall i\ (o_1 \succ_i' o_2 \Leftrightarrow o_1 \succ_i'' o_2) \Rightarrow (o_1 \succ_{W([\succ'])} o_2 \Leftrightarrow o_1 \succ_{W([\succ''])} o_2)$

The absence of a dictator

We say that W is not a dictator, if there is no such i, that $\forall o_1, o_2 \in O\ (o_1 \succ_i o_2 \Rightarrow o_1 \succ_W o_2)$

Arrow's theorem

If $|O| \geq 3$, then any Pareto efficient, independent of irrelevant alternatives social welfare function W is a dictator.

The proof proceeds in four steps.

Step 1.

If each agent puts the outcome of b in the top or the bottom of your list of preferences, hence in ≻W the b outcome will also be either at the top or bottom of the list.

Take an arbitrary profile [≻] such that in it for all agents i the b outcome is either at the top or bottom of the list of preferences ≻i. Now assume that the assertion is false, that there exist a, c ∈ O, such that a ≻W b and b ≻W c. Then change the profile [≻], so that all agents satisfy c ≻i a, without changing the ranking of the other outcomes. We denote the resulting profile [≻ ']. Since the outcome after the modification b for each agent would still be either on the upper or lowermost position on the list of his preferences, the independence of extraneous W alternatives can conclude that a new profile and a ≻W b and b ≻W c. Consequently, by the transitive ≻W get a ≻W c. But we have assumed that all agents c ≻i a, then by Pareto efficiency should be c ≻W a. This contradiction proves the claim.

Step 2.

There exists an agent which is central in the sense that changing the voice, i. e. he can move an outcome from b from lowermost position in ≻W list to the uppermost position in the list.

Consider any preference profile in which all agents have arranged outcome b at the bottom of their list of preferences ≻i. It is clear that in ≻W outcome b is at the bottom position. Let all agents took turns to rearrange the outcome of b from the lowest to the topmost position in their list of preferences, without changing the ranking of the other outcomes. Let n* - an agent that after putting so b, changed ≻W. We denote [≻1] - a preference profile just before the n* moved b, a [≻2] - profile of preferences as soon as n* moved

b. Thus, in [≻2] b outcome changed its position in ≻W, wherein for all agents b is either the uppermost or the lowermost position ≻i. Consequently, by the assertion proved in Step 1, in the outcome b ≻W occupies the top position.

Step 3.

n * is dictator over all pairs , which not include b.

Choose from a pair of any item. Without loss of generality, we choose a.

Next, from the profile [≻2] build [≻3] as follows: in ≻n* move the outcome a to the first position, leaving the rest unchanged ranking, arbitrarily for all other agents interchange with each other a and c. Then, as in [≻1] we find that a ≻W b (because of the independence of irrelevant alternatives), and, as in [≻2] we find that b ≻W c. Then a ≻W c. Now we construct a preference profile [≻4] as follows: for all agents put the outcome b to an arbitrary position in the list of preferences ≻i, for n* agent put the outcome a in a random position before outcome c. Clearly, in view of the independence of irrelevant alternatives a ≻W c. We got that all agents except n* have completely arbitrary profiles of preferences, and the result a ≻W c was derived solely from the assumption that a ≻n* c.

Step 4.

n* is dictator over all pairs .

Consider any outcome c. Since Step 2, there is some n**- central agent for this outcome, he is a dictator for all pairs , where, in particular, A = a, B = b. But n* himself may change the rankings in ≻W (this was seen in Step 2). Therefore, we can conclude that n* coincides with n**. The proof is complete.

Note: Arrow’s Impossibility Theorem has a particular case – Condorcet’s paradox. It lies in the fact that the presence of more than two alternatives, and more than two voters collective ranking of alternatives can be cyclical (non-transitive), even if the rankings of all voters are cyclical (transitive). Thus, the expression of different groups of voters, each of which represents a majority, may enter into a paradoxical contradiction with each other.

Works Cited

Campbell, D. E. and Kelly, J. S. (2002) Impossibility theorems in the Arrovian framework, in Handbook of social choice and welfare (ed. by Kenneth J. Arrow, Amartya K. Sen and Kotaro Suzumura), volume 1, pages 35–94, Elsevier.

The Mathematics of Behavior by Earl Hunt, Cambridge University Press, 2007. The chapter " Defining Rationality: Personal and Group Decision Making" has a detailed discussion of the Arrow Theorem, with proof. URL to CUP information on this book

Why flip a coin? : the art and science of good decisions by Harold W. Lewis, John Wiley, 1997. Gives explicit examples of preference rankings and apparently anomalous results under different voting systems. States but does not prove Arrow's theorem. ISBN 0-471-29645-7

Sen, A. K. (1979) “ Personal utilities and public judgements: or what's wrong with welfare economics?” The Economic Journal, 89, 537-558

Yu, Ning Neil (2012) A one-shot proof of Arrow's theorem. Economic Theory, volume 50, issue 2, pages 523-525, Springer.