Example of definition of a group term paper

Sociology, Community

A group is a non empty set G with a binary operation, * defined on it and so that the following axioms hold.

G1: a *(b*c) = (a*b)*c for all a, b, c in G

G2: there is an element 1 in G such that 1*a = a*1 = a for all in a in G

G3: for every element a in G, there is an element a-1 in G such that a*a-1 = 1 = a-1*a

Any group is an abelian group if for a, b in G then a*b = b*a. it is very common to write abelian group operations with additive operation because many of the groups arise from additive structures in number systems like integers, real numbers. We can therefore define an abelian group with the above axioms including a fourth as follows;

Definition: An abelian is a nonempty set A with binary operation with a binary operation, + defined on set A with the following axioms

A1. Associativity property: for all a, b, c in A, then a+ (b + c) = (a + b) +c

A2. Commutative property: for all a, b in A then a + b = b + a

A3. Additive identity: there is an element 0 in A so that 0 + a = a, for all a in A

A4. Additive inverse: for every element a in A, there is an element –a in A such that –a + a = a –a = 1.

We note here that any field is an abelian group under addition. Under multiplication, the nonzero elements of any field must also form an abelian group. These results are connected by the distributive laws.

The Abelian group definition above is very useful in the discussion of vector spaces and modules. We can also define a vector space to be an abelian group together with multiplication to satisfy the relevant properties. We can therefore state the definition of a module by using the definition of vector spaces in the following way

Definition: A left module over a ring R is an abelian group A together with scalar multiplication * defined on A for the conditions below to stand. For all a, b, in R and x, y in A the following conditions hold;

1. a*x in A
2. a*(b*c) = (ab)*x
3. (a + b)*x = a*x + b*x
4. a*(x + y) = a*x + a*y
5. 1 * x = x

A homomorphism is a map between any two abelian groups that satisfies the condition. When we write additively, we write

Also any one to one homomorphism is also called isomorphism and any two abelian groups are isomorphic if there is an isomorphism before the two groups.

We can an example of this is that the character of A is a homomorphism from A to the multiplicative group of non- zeros complex numbers.

Finite Abelian Groups: We commonly use additive notation when dealing with abelian groups and when we write them like this we call them additive group. Some properties are summarized in the table below.

Definition : If A, B are additive groups, then direct sum AB is a Cartesian product A B which is also additive group under the coordinate addition (a1, b1) + (a2, b2) = (a1 + b1, a2 + b2)

Theorem: Suppose A, B are subgroups of an additive group C which satisfy the conditions

(1). A B = 0. (2). A + B = C (A + B = {a + b | a}). Then C is said to be isomorphic to the direct sum AB. the above theorem brings us to conclude that C is an internal direct sum of A and B or otherwise C = A B.

We have the proof of the above using the first isomorphic theorem.

Let F: A B is given by f (a, b) = a + b

1. F is homomorphism. F ((a1, b1), (a2, b2)) = f (a1+a2, b1+b2) = a1+a2+b1+b2

F (a1, b1) +F (a2, b2) = a1+b1+a2+b2. These are equal since the addition is commutative in C

2. Ker F = {(a, b) | F (a, b) = a + b= 0}. a + b = 0 implies that a = -b found in AB = 0 so a = b = 0 and Ker F = 0

3. Im F = {a +b | a} = A +B = C

Fundamental theorem of finite abelian groups

In this theorem, we shall look at some basics. These are cyclic groups and direct sums

Cyclic Groups: these are groups in which the elements are of the form where n is the smallest positive integer and a is called group generator which satisfies the condition na = 0.

Direct Sum: Given two groups A, and B, with, and, the direct sum is the set of pairs (a, b) with operation as a rule that. The zero of this Abelian group is (0, 0) and its inverse is given by (-a, -b). If this group is written in multiplicative form, then we talk about the direct product which is written as.

We can thus state the fundamental theorem of finite groups in two ways;

Theorem One

Any finite abelian group is isomorphic to a direct sum of cyclic groups. This emanates from the fact that two direct sums may be isomorphic. For instance for two groups A2, A3 and A6, the relation is. And if a and b are generators of summands, then. The progressive multiples of the pair (a, b) are given by (0, 0), (0, b), (a, 0), (a, b), (0, 2b) and (a, 2b).

Any such problem has basically two standard solutions which can be noted as;

(i). Note that an abelian group is in smith canonical form if it is written in the form of where are integers greater than unity and are multiples of and

(ii). An abelian group is in prime power canonical form if it is written in the form of where and are the prime powers.

Theorem Two

We can state this in two ways;

(i) Any finite abelian group can be written in smith canonical form. We should as well note that if two groups which are in smith canonical form are isomorphic, and then the multi- sets of orders of the cyclic factors are equal.

(ii) We can convert an abelian group from the smith into prime-power and vise versa. To convert smith canonical into prime power canonical, we use the fact that if and are powers of the prime numbers then we write. In this case we can simply do factorization of the orders of cyclic factors.

Whereas if we are to convert the power prime to smith canonical form, we need to gather up the largest power of each prime power and we multiply them. We repeat the iterations until it is simplified to the smallest power. An example of these below

The smith canonical form is. The above group can be written in the prime-power canonical form as. These groups are isomorphic.