Euler’s formula essay sample

Science, Mathematics

The 18 th century mathematician Leonard Euler is considered a pioneering mathematician and physicist. He is remembered for his contributions to calculus and graph theory, many of which bear his name. Some of the concepts attributed to Euler include Euler cycles and paths in graph theory, Euler equations in fluid dynamics, Euler-Maclaurin formula in calculus, Euler’s number, among other things. Called the Mozart of Mathematics, Euler was able to publish 800 papers and books on both pure and applied mathematics during his lifetime. As a point of comparison, modern mathematicians have an average lifetime output of twenty papers.

Due to his mathematical tenacity, people who are familiar with the breadth and depth of Euler’s work put him at par with Newton, Maxwell, Gauss and other mathematical prodigies. A survey by Physics World in 2004 asked their readers what they considered to be the most beautiful equation in Mathematics. The top twenty equations had two equations named after Euler, including the magnificent Euler formula. A formula which noted theoretical physicist Richard Feynmann referred to as “[our] jewel” and “ the most remarkable formula in mathematics”.

Euler’s Formula

Euler’s famous formula is given by the following:

One thing to note about Euler’ Formula is how it manages to combine and show the connections between several key mathematical quantities. In one simple line, the formula gives an elegant relationship between the mathematical constant (e), the imaginary number *i* . and the two basic trigonometric identities of sine and cosine.

As such, Euler's formula ties together the areas of mathematical analysis, trigonometry, and complex number theory. Moreover, when x = π, we get a special case of Euler's formula – the Euler identity.

Moreover, using Euler's formula, we can get an expression for sine and cosine which uses the exponential function. Adding and subtracting Euler's formula evaluated at x with Euler's formula evaluated at –x gives us the following forms for sine and cosine.

History of Euler's Formula

Euler made the first mention of his formula in his 1748 book *Introductio in analysin infinitorum.* His proof relied on showing that the infinite series expansion of both sides were equal. However, it wasn't Euler who first made mention of the formula. English mathematician Roger Cotes first gave the proof for the formula in 1714. However, Cotes gave the formula in the following form (Stillwell).

However, both Euler and Cotes didn't see the geometrical interpretation of the formula wherein complex numbers are seen as points in a complex plane. That idea would have to wait until the work of Danish-Norwegian mathematician Caspar Wessel. His 1799 paper *Om directionens analytiske betegning* is universally recognized as the origin for the ideas of a complex plane (" Caspar Wessel").

How the formula is used

A key concept to grasp in understanding Euler’s formula is the aforementioned complex plane. The complex plane is a geometric representation of complex numbers. It is similar to the Cartesian plane. Complex numbers are given as an ordered pair consisting of their real and imaginary parts. The value of the real version translates to a displacement along the horizontal axis while the magnitude of the imaginary part translates to a displacement along the vertical axis. In this case, the horizontal axis becomes the real axis and the vertical axis becomes the imaginary axis. Additionally, with the complex plane, complex numbers can also be defined as the magnitude of a vector as well as a corresponding angular displacement from the positive real axis. The convention for the angle is positive direction is counter clockwise from the positive real axis.

Representation of Complex Numbers in the Complex plane

With the complex plane, operations on complex numbers can be viewed geometrically. Addition of complex numbers becomes similar to addition of vectors in the plane. Multiplication of complex numbers is best visualized using the angular definition. The product of two complex numbers is a complex number with a magnitude equal to the product of the magnitudes of the two multipliers and its angle is equal to the sum of the angles of the two complex multipliers.

With the complex plane, we can represent complex numbers as an ordered pair or as a magnitude and angle. Euler’s formula adds another representation of complex numbers – as complex exponentials. Looking back at Euler's formula . We see that the quantity e^{ix} is a complex number with a real part with a magnitude of cos x and an imaginary part with a value of sin x. Plotting this on the complex plane, we can easily verify that a complex number with the following real and imaginary parts has an angle equal to x. Therefore, through Euler's formula, we can represent complex numbers as $|Z| e^{ix}$ where A represents their magnitude and x represents their angle.

The major benefit of the exponential form is the ease by which we can analytically multiply complex numbers. If we have two complex numbers with magnitudes Z_1 and Z_2 and corresponding angles A_1 and A_2, their product would have a magnitude of $Z_1 Z_2$ and angle of $(A_1 + A_2)$. If we perform this using the exponential form, our two multipliers would be $Z_1 e^{iA1}$ and $Z_2 e^{iA2}$. Performing the multiplication using conventional algebra gives us the product $Z_1 Z_2 e^{i(A1+A2)}$ which is the valid product for our complex multiplication.

So far, we have looked at the complex exponential as a representation for complex numbers. Another mathematical concept that the complex exponential represents via the Euler formula are the time varying trigonometric functions of sine and cosine. We do this by adding a parametric multiplier to our complex exponential. This multiplier is usually time as in e^{ixt}. Looking closely at the expression e^{ixt}, it represents a complex point whose angle is increasing as time goes on. In a sense, it is rotating around the origin. With the right hand of Euler's formula, we know that this rotating complex number’s projection along the real axis is cos xt. Similarly, its projection along the imaginary axis is isin xt. Put another way, cos xt is simply Re[e ^{ixt}]. Therefore, we have another representation for sinusoids – the complex exponential. Similar to complex numbers, the beauty of using complex exponentials as representation for sinusoids is that they make analytical operations on sinusoids easier. Multiplication of sinusoids reduces to adding the exponents of two complex exponentials. The ease by which the complex exponential helps in representing and manipulating sinusoids is most apparent when you try to give the proof for several trigonometric identities using the complex exponential as in the following example:

Applications of Euler’s Formula

One major application of Euler’s formula is in frequency analysis. The Fourier transform allows for the breakdown of any mathematical waveform into a summation of sinusoids of different frequencies and amplitudes. It is here that the Euler formula is useful as it reduces the two Fourier analysis equations into one. Using the complex exponential form of the Euler formula allows us to solve for the odd (sine) and even (cosine) components of our target signal in a single pass. The same is also true for the synthesis part, as using the complex exponential form leaves us with only one set of coefficients instead of two is we used the sinusoid forms. This simplification of Fourier analysis via Euler’s formula is used in many different fields, from boundary value problems in various fields of engineering, to understanding various forms of resonance and waves in physics to understanding the spectral qualities of signals in signal processing.

Indeed, Euler’s reconciliation of complex exponentials with sinusoids is best appreciated in fields dominated oscillations and waves as these phenomena are described by sinusoidal functions. In electrical engineering, the phasor concept is used in simplifying the analysis of alternating current and power. Communications engineering uses the complex exponential as a way to simplify the analysis of the various forms of continuous wave modulation which is the basis of nearly all modern wireless communications. Altogether, these fields are responsible for much of the features of modern civilization we all know and enjoy. All of this stemming from the Mozart of Mathematics’ elegant marriage of complex numbers, the exponential and the trigonometric functions.

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