Construction of real numbers

All mathematicians know (or think they know) all about the real numbers. However usually we just accept the real numbers as ‘ being there’ rather than considering precisely what they are. In this project I will attempts to answer that question. We shall begin with positive integers and then successively construct the rational and finally the real numbers. Also showing how real numbers satisfy the axiom of the upper bound, whilst rational numbers do not. This shows that all real numbers converge towards the Cauchy’s sequence.

1 Introduction

What is real analysis; real analysis is a field in mathematics which is applied in many areas including number theory, probability theory. All mathematicians know (or think they know) all about the real numbers. However usually we just accept the real numbers as ‘ being there’ rather than considering precisely what they are. The aim of this study is to analyse number theory to show the difference between real numbers and rational numbers.

Developments in calculus were mainly made in the seventeenth and eighteenth century. Examples from the literature can be given such as the proof that π cannot be rational by Lambert, 1971. During the development of calculus in the seventeenth century the entire set of real numbers were used without having them defined clearly. The first person to release a definition on real numbers was Georg Cantor in 1871. In 1874 Georg Cantor revealed that the set of all real numbers are uncountable infinite but the set of all algebraic numbers are countable infinite.

As you can see, real analysis is a somewhat theoretical field that is closely related to mathematical concepts used in most branches of economics such as calculus and probability theory. The concept that I have talked about in my project are the real number system.

2 Definitions

Natural numbers

Natural numbers are the fundamental numbers which we use to count. We can add and multiply two natural numbers and the result would be another natural number, these operations obey various rules.

(Stirling, p. 2, 1997)

Rational numbers

Rational numbers consists of all numbers of the form a/b where a and b are integers and that b ≠ 0, rational numbers are usually called fractions. The use of rational numbers permits us to solve equations. For example; a + b = c, ad = e, for a where b, c, d, e are all rational numbers and a ≠ 0. Operations of subtraction and division (with non zero divisor) are possible with all rational numbers.

(Stirling, p. 2, 1997)

Real numbers

Real numbers can also be called irrational numbers as they are not rational numbers like pi, square root of 2, e (the base of natural log). Real numbers can be given by an infinite number of decimals; real numbers are used to measure continuous quantities. There are two basic properties that are

involved with real numbers ordered fields and least upper bounds. Ordered fields say that real numbers comprises a field with addition, multiplication and division by non zero number. For the least upper bound if a non empty set of real numbers has an upper bound then it is called least upper bound.

Sequences

A Sequence is a set of numbers arranged in a particular order so that we know which number is first, second, third etc… and that at any positive natural number at n; we know that the number will be in nth place. If a sequence has a function, a, then we can denote the nth term by an. A sequence is commonly denoted by a1, a2, a3, a4… this entire sequences can be written as or (an). You can use any letter to denote the sequence like x, y, z etc. so giving (xn), (yn), (zn) as sequences

We can also make subsequence from sequences, so if we say that (bn) is a subsequence of (an) if for each n∈ â„• we get;

bn = ax for some x ∈ â„• and bn+1 = by for some y ∈ â„• and x > y.

We can alternatively imagine a subsequence of a sequence being a sequence that has had terms missing from the original sequence for example we can say that a2, a4 is a subsequence if a1, a2, a3, a4.

A sequence is increasing if an+1 ≥ an ∀ n ∈ â„•. Correspondingly, a sequence is decreasing if

an+1 ≤ an ∀ n ∈ â„•. If the sequence is either increasing or decreasing it is called a monotone sequence.

There are several different types of sequences such as Cauchy sequence, convergent sequence, monotonic sequence, Fibonacci sequence, look and see sequence. I will be talking about only 2 of the sequences Cauchy and Convergent sequences.

Convergent sequences

A sequence (an) of real number is called a convergent sequences if an tends to a finite limit as n→∞. If we say that (an) has a limit a∈ F if given any ε > 0, ε ∈ F, k∈ â„• | an – a | < ε n ≥ k

If an has a limit a, then we can write it as liman = a or (an) → a.

Cauchy Sequence

A Cauchy sequence is a sequence in which numbers become closer to each other as the sequence progresses. If we say that (an) is a Cauchy sequence if given any ε > 0, ε ∈ F, k∈ â„• | an – am | < ε n, m ≥ k.

Gary Sng Chee Hien, (2001).

Bounded sets, Upper Bounds, Least Upper Bounds

A set is called bounded if there is a certain sense of finite size. A set R of real numbers is called bounded of there is a real number Q such that Q ≥ r for all r in R. the number M is called the upper bound of R. A set is bounded if it has both upper and lower bounds. This is extendable to subsets of any partially ordered set. A subset Q of a partially ordered set R is called bounded above. If there is an element of Q ≥ r for all r in R, the element Q is called an upper bound of R

3 Real number system

Natural Numbers

Natural numbers (â„•) can be denoted by 1, 2, 3... we can define them by their properties in order of relation. So if we consider a set S, if the relation is less than or equal to on S

For every x, y ∈ S x ≤ y and/or y ≤ x

If x ≤ y and y ≤ x then x = y

If x ≤ y and y ≤ z then x ≤ z

If all 3 properties are met we can call S an ordered set.

(Giles, p. 1, 1972)

Real numbers

Axioms for real numbers can be spilt in to 3 groups; algebraic, order and completeness.

Algebraic Axioms

For all x, y ∈ â„, x + y ∈ â„ and xy ∈ â„.

For all x, y, z ∈ â„, (x + y) + z = x (y + z).

For all x, y ∈ â„, x + y = y + x.

There is a number 0 ∈ â„ such that x + 0 = x = 0 + x for all x ∈ â„.

For each x ∈ â„, there exists a corresponding number (-x) ∈ â„ such that x + (-x) = 0 = (-x) + x

For all x, y, z ∈ â„▯, (x y) z = x (y z).

For all x, y ∈ â„▯ x y = y x.

There is number 1 ∈ â„▯ such that x x 1 = x = 1 x x, for all x ∈ â„▯

For each x ∈ â„▯ such that x ≠ 0, there is a corresponding number (x-1) ∈ â„▯ such that x (x-1) = 1 = (x-1) x

A10. For all x, y, z ∈ â„▯, x (y + z) = x y + x z

(Hart, p. 11, 2001)

Order Axioms

Any pair x, y of real numbers satisfies precisely one of the following relations: (a) x < y; (b) x = y; (c) y < x.

If x < y and y < z then x < z.

If x < y then x + z < y +z.

If x < y and z > 0 then x z < y z

(Hart, p. 12, 2001)

Completeness Axiom

If a non-empty set A has an upper bound, it has a least upper bound

The thing which distinguishes â„▯ from is the Completeness Axiom.

An upper bound of a non-empty subset A of R is an element b ∈R with b a for all a ∈A.

An element M ∈ R is a least upper bound or supremum of A if

M is an upper bound of A and if b is an upper bound of A then b M.

That is, if M is a least upper bound of A then (b ∈ R)(x ∈ A)(b x) b M

A lower bound of a non-empty subset A of R is an element d ∈ R with d a for all a ∈A.

An element m ∈ R is a greatest lower bound or infimum of A if

m is a lower bound of A and if d is an upper bound of A then m d.

If all 3 axioms are satisfied it is called a complete ordered field.

John o'Connor (2002) axioms of real numbers

Rational numbers

Axioms for Rational numbers

The axiom of rational numbers operate with +, x and the relation ≤, they can be defined on corresponding to what we know on N.

For on +(add) has the following properties.

For every x, y ∈ , there is a unique element x + y ∈

For every x, y ∈ , x + y = y + x

For every x, y, z ∈ , (x + y) + z = x + (y + z)

There exists a unique element 0 ∈ such that x + 0 = x for all x ∈

To every x ∈ there exists a unique element (-x) ∈ such that x + (-x) = 0

For on x(multiplication) has the following properties.

To every x, y ∈ , there is a unique element x x y ∈

For every x, y ∈ , x x y = y x x

For every x, y, z ∈ , (x x y) x z = x x (y x z)

There exists a unique element 1 ∈ such that x x 1 = x for all x ∈

To every x ∈ , x ≠ 0 there exists a unique element ∈ such that x x = 1

For both add and multiplication properties there is a closer, commutative, associative, identity and inverse on + and x, both properties can be related by.

For every x, y, z ∈ , x x (y + z) = (x x y) + (x x z)

For with an order relation of ≤, the relation property is <.

For every x ∈ , either x < 0, 0 < x or x = 0

For every x, y ∈ , where 0 < x, 0 < y then 0 < x + y and 0 < x x y

For every x, y ∈ , x < y if 0 < y – x

(Giles, pp. 3-4, 1972)

From both the axioms of rational numbers and real numbers, we can see that they are about the same apart from a few bits like rational numbers do not contain square root of 2 whilst real numbers do. Both rational and real

numbers have the properties of add, multiplication and there exists a relationship of 0 and 1.

4 Proofs

In this section I will be solving some basic proofs, most of my proofs have been assumed in the construction process and have been reduced.

Theorem:

Between any two real numbers is an rational number.

Proof

Let a ≠ b be a real number with a < b. so if we choose n so that it is . Then we can look at the multiples of. Since these are not bounded in any way we may choose the first multiple as > a. we can claim that < b. if not then since < a and > b we would have > b – a.

John O'Connor (2002) axioms of real numbers

Theorem:

The limit of a sequence, if it exists, is unique.

Proof

Let x and x′ be 2 different limits. We may assume without loss of generality, that

x < x′. In particular, take ε = (x′ – x)/2 > 0.

Since xn→ x, k1 s. t

$| xn – x | < n \geq k1$

Since $xn \rightarrow x$ k2 s. t

$| xn – x'| < \varepsilon$ $n \geq k2$

Take $k = \max\{k1, k2\}$. Then $n \geq k$,

$| xn – x | < \varepsilon$, $| xn – x'| < \varepsilon$

$| x' – x | = | x' – xn + xn – x |$

$\leq | x' – xn | + | xn – x |$

$< \varepsilon + \varepsilon$

$= x' – x$, a contradiction!

Hence, the limit must be unique. Also all rational number sequences have a limit in real numbers.

Gary Sng Chee Hien, (2001).

Theorem:

Any convergent sequence is bounded.

Proof

Suppose the sequence (an)®a. take = 1. Then choose N so that whatever n > N we have an within 1 of a. apart from the finite set {a1, a2, a3...aN} all the terms of the sequence will be bounded by a + 1 and a – 1. Showing that

an upper bound for the sequence is max{a1, a2, a3...aN, a +1}. Using the same method you could alternatively find the lower bound

Theorem:

Every Cauchy Sequence is bounded.

Proof

Let (xn) be a Cauchy sequence. Then for

| xn – xm | < 1 n, m ≥ k.

Hence, for n ≥ k, we have

| xn | = | xn – xk + xk |

≤ | xn – xk | + | xk |

< 1 + | xk |

Let M = max{ | x1 |, | x2 |, ..., | xk-1|, 1 + | xk | } and it is clear that | xn | ≤ M n, i. e. (xn) is bounded.

Gary Sng Chee Hien, (2001).

Theorem:

If (xnx, then any subsequence of (xn) also converges to x.

Proof

Let (yn) be any subsequence of (xn). Given any > 0, s. t

| xn – x | < n ≥ N.

But yn = xi for some so we may claim

| yn – x | < also.

Hence, (

Gary Sng Chee Hien, (2001).

Theorem:

If (xn) is Cauchy, then any subsequence of (xn) is also Cauchy.

Proof

Let (yn) be any subsequence of (xn). Given any s. t

| xn – xm | .

But yn = xi for so we may claim

| yn – ym |

Hence (yn) x

Gary Sng Chee Hien, (2001).

Theorem

Any convergent sequence is a Cauchy sequence.

Proof

If (an) a then given > 0 choose N so that if n > N we have | an- a| < . Then if m, n > N we have | am- an| = |(am- a) – (am- a)| | am- a| + | am- a| < 2.

We use completeness Axiom to prove

Suppose X ∈ â„▤, X2 = 2. Let (an) be a sequence of rational numbers converging to an irrational

12 = 1

1. 52 = 2. 25

1. 42 = 1. 96

1. 412 = 1. 9881

1. 41421356237302 = 1. 999999999999731161391129

Since (an) is a convergent sequence in â„▤ it is a Cauchy sequence in â„▤ and hence also a Cauchy sequence in . But it has no limit in.

An irrational number like 2 has a decimal expansion which does not repeat:

2 = 1. 4142135623730

John O'Connor (2002) Cauchy Sequences.

Theorem

Prove that is irrational, prove that ≤ â„▤

Proof

We will get 2 as the least upper bound of the set A = {q Q | q2 < 2}. We know that a is bounded above and so its least upper bound b does not exists.

Suppose x ∈ , x2 <2, look at (x +)2 < 2

(x +)2

< 2

So if we pick then (x +)2 < 2

So x is not an upper bound of A. This shows least upper bound x cannot satisfy x2 <2. From this we can choose a n to satisfy the condition. Leading to the conclusion that x would not be an upper bound. Proving that 2 is irrational. Therefore proves that ≤ â„▤.

Solving using the Newton's method

xn+1 = (xn+ 2/xn)/2 and x1 = 1.

This gives (1 , 3/2 , 17/12 , 577/408 , 665857/470832 , …) which is approximately (1, 1. 5, 1. 41667, 1. 414215, 1, 414213562, …)

John O'Connor (2002) Convergence in the real's.

Theorem:

Let (xn) and (yn) be two Cauchy sequences. Then the following holds:

(i) (xn + yn) is Cauchy.

(ii) (xn yn) is Cauchy.

Proof

(i) Let any $\varepsilon > 0$ be given. Then k1, k2 s. t

$| xn – xm | < \varepsilon/2$ $n \geq k1$

$| yn – ym | < \varepsilon/2$ $n \geq k2$

Take k = max(k1, k2). Then

$| xn – xm | < \varepsilon/2$, $| yn – ym | < \varepsilon/2$ $n \geq k$.

But

$| (xn + yn) – (xm + ym) | = | (xn – xm) + (yn – ym) |$

$\leq | xn – xm | + | yn – ym |$

$< \varepsilon/2 + \varepsilon/2$ $n \geq k$.

$= \varepsilon$ $n \geq k$.

Hence, (xn + yn) is also Cauchy.

(ii) Now, since (xn), (yn) is Cauchy, they are bounded by some X, Y ≠ 0. Let any

$\varepsilon > 0$ be given. Then k1, k2 s. t

$| xn – xm | < \varepsilon/(2Y)$ $n, m \geq k1$

$| yn – ym | < \varepsilon/(2X)$ $n, m \geq k2$

Take k = max(k1, k2). Then

$| x_n - x_m | < \varepsilon/(2Y)$

$| y_n - y_m | < \varepsilon/(2X)$ $n, m \geq k$

Hence,

$| x_n y_n - x_m y_m | = | (x_n y_n - x_m y_n) + (x_m y_n - x_m y_m) |$

$\leq | x_n y_n - x_m y_n | + | x_m y_n - x_m y_m |$

$= | y_n | \, | x_n - x_m | + | x_m | \, | y_n - y_m |$

$\leq Y | x_n - x_m | + X | y_n - y_m |$

$< Y(\varepsilon/(2Y)) + X(\varepsilon/(2X))$ $n, m \geq k$

$=$

Hence, $(x_n y_n)$ is also Cauchy.

5 Conclusion

Real numbers are infinite number of decimals used to measure continuous quantities. On the other hand, rational numbers are defined to be fractions formed from real numbers. Axioms of each number system are examined to determine the difference between real numbers and rational numbers. Conclusion of the analysis of axioms resulted to be both real numbers and rational numbers contain the same properties. The properties being addition, multiplication and there exist a relationship of zero and one.

The four fundamental results are obtained from this study. First concept is that the property of real number system being unique and following the complete ordered field. Second is that if any real number satisfies the

axioms then it is upper bound, whilst rational numbers are not upper bound. The third being that all Cauchy sequences are converges towards the real numbers. Finally found out that all real numbers are equivalence classes of the Cauchy sequence.

Appendices

List of symbols

â„• = Natural number

â„ = Real number

= Rational number

∈ = is an element of

= There exists

= For all

s. t. = Such that