What is mathematics pedagogy?

A necessary premise to the question What is mathematics pedagogy. is Why do we teach mathematics. If basic number skills are obviously needed in every days life, can we say the same about geometry? Mathematics trains us to think and to think rationally, rigorously. They are an essential part of human culture and a heritage of the past and anyone who studied mathematics seriously found profound beauty in them. More important mathematics helps us understand, since Galileo we know they help us understand the science of nature, but also the relationship between the vertices, edges and faces of a polyhedron or that any integer greater than 1 can be expressed as a unique (up to ordering) product of prime numbers…

To the question why doing mathematics David Hilbert replied: " The problem is here, you must solve it!" However the answer that would probably convince most is simply because mathematics is useful: in sciences (physics, chemistry, biology, computer science…), in engineering, in economics and finance… And except if one wants to suppress every form of teaching, we have to teach mathematics. Solid mathematics education is imperative for broadening post-school opportunities. Unfortunately, many students progress are hampered by lack of confidence, or low numeracy skills and it is for teachers to improve their practice to crack the pattern. According to the International Academy of Education, a multinational organism working with the UNESCO, any mathematical pedagogy should abide to the following principles:

- be grounded in the general premise that all students have the right to access education and the specific premise that all have the right to access mathematical culture;

- acknowledge that all students, irrespective of age, can develop positive mathematical identities and become powerful mathematical learners;

- be based on interpersonal respect and sensitivity and be responsive to the multiplicity of cultural heritages, thinking processes, and realities typically found in our classrooms;

- be focused on optimising a range of desirable academic outcomes that include conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning;

- be committed to enhancing a range of social outcomes within the mathematics classroom that will contribute to the holistic development of students for productive citizenship.

(International Academy of Education, p6)

This essay is divided into three parts. First we give some considerations about mathematics, what they are and what are the distinctions from other sciences. The second part is a broad look at the notion of pedagogy and the differences between traditional and progressive pedagogy. Finally in the third we will concentrate on mathematical pedagogy and the two ways of looking at mathematical concepts, object or tool.

What is Mathematics?

Mathematic comes from the Greek Î¼Î¬Î¸Î·Î¼Î± (máthÄ“ ma) whose first meaning was “ science or knowledge”, and only later became “ mathematics”. From MáthÄ“ ma came Î¼Î±Î¸Î·Î¼Î±Ï„ Î¹ÎºÏŒÏ, (mathematikos), which first meant “ related to learning”, and then later “ related to the mathematical science”. In Latin it became mathematicus then mathematique in Romance languages. Finally mathematic then mathematics.

The Ishango bone is the oldest trace of mathematical activity discovered; dating back to the prehistoric ages, it is estimated to be more than 20000 years old (Brooks and Smith, 1987). In comparison writing is much more recent, dated between 6000 BC and 8000 BC.

However mathematics as we understand it today started with the Ancient Greeks between 600 and 300 BC. If all culture and civilization necessarily developed their knowledge of mathematics, they all did it through inductive reasoning: one reaches conclusions or rules through repeated observations. On the other hand, the Greek mathematicians used deductive reasoning: they used definitions, axioms and logic to prove their statements (Bernal, 2000). The most influential Mathematic text of all time, the Elements was written in 300 BC in Alexandria by Euclid (Boyer and Merzbach, 1991) and was used as a textbook until the end of the nineteenth century. This shows one of the most characteristic feature of Mathematics, the permanence of its results. The history of Mathematics or even a summary of it is far beyond the scope of this essay, we will shortly present two moments of the history of Mathematics, which are of relevance for this essay. In The Assayer, Galileo wrote “ Philosophy is written in this grand book, the universe … It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures;….” (Drake and Stillman, 1957). Galileo stemmed the mathematization of science that gave birth to the science we know today. The importance of present Mathematics, the fact that Mathematics support most sciences and techniques, that good education in Mathematics is of primordial importance in composing autonomous citizens roots from this momentous time.

Modern Mathematics ushered in the early 1920s from the foundational crisis of mathematics. David Hilbert (1862-1943) proposed a refundation of Mathematics, known as the Hilbert's program. Mathematics was to be written in a formal language and rooted in a set of axioms. Already in the Element, Euclid based his work on a few axioms and postulates. Those were seen as obvious propositions on existing objects. However Hilbert's axioms are not supported by existing or abstract objects, they are a set of relations and operations. This new approach shaped the modern mathematics, and connectively how they are taught.

Mathematics is a science in the traditional sense, i. e. an organized body of knowledge characterized by the choice of the objects studied and the inner rules of formation and validation of its knowledge. With Galileo came the idea that the law of nature could be written mathematically and because of its phenomenal success, modern science extensively use mathematics; constant efforts are made to write any scientific rule in the mathematical language. However Mathematics is not a modern science nor a science of nature.

Defining mathematics is a complicated task and a work in progress. Mathematicians had and still have heated debates and there is no satisfying definition (Mura, 1993).

One of the first definition was given by Aristotle: “ the science of quantity” (Franklin, 2009). Where arithmetic was the field of discrete quantity and geometry the field of the continuous ones. A more modern definition was given by the American mathematician Benjamin Peirce, the first line of his Linear Associative Algebra is “ Mathematics is the science that draws necessary conclusions”. Logicism (i. e. adequacy of mathematics and logic) challenged this definition, for example Bertrand Russel wrote “ All Mathematics is Symbolic Logic”. (Russell, 1903). In turn this view was criticized by the intuitionism mathematicians who saw mathematics as a mental activity constructing objects: “ The truth of a mathematical statement can only be conceived via a mental construction that proves it to be true, and the communication between mathematicians only serves as a means to create the same mental process in different minds”(Iemhoff, 2012).

In order to give a definition of mathematics one needs to first precise its field of study and second explain its mechanism.

One of the most talented mathematician of the twentieth century, Alexander Grothendieck wrote in Récoltes et semailles “ mathematical activity involves essentially three things: studying numbers, studying shapes and measuring distances.” Arithmetic studies numbers, geometry shapes and analysis distances. However if Mathematics is a single science and not three distinct ones, it is because there are tight connections between numbers, shapes and distances and studying those connections takes a great part of the mathematics.

The rules of mathematics are the most constraining of all sciences. For a notion to be mathematic, its definition needs to contain it entirely. In everyday language, words are understood because they relate to multiple personal experiences. In some sense the definition of a word only encapsulate part of its meaning, words don't fully contain the notions they designate. In mathematics this aspect of language is prohibited, a word can only be used if its definition determines its whole meaning, and this, only using words whose definition are already established. Definitions must hold by themselves without any exterior references. Moreover the rules defining the relation between words, i. e., the rules of logic, must also be completely explicit. A mathematical text is about well-defined objects and shows some necessary consequences of their definitions through a combination of logical steps.

What is pedagogy?

" Let this variety of ideas be set before him; he will choose if he can; if not, he will remain in doubt. Only the fools are certain and assured. For if he embraces Xenophon's and Plato's opinions by his own reasoning, they will no longer be theirs, they will be his. He who follows another follows nothing. He finds nothing; indeed he seeks nothing. We are not under a king; let each one claim his own freedom [Seneca]. Let him know that he knows, at least. He must imbibe their ways of thinking, not learn their precepts. And let him boldly forget, if he wants, where he got them, but let him know how to make them his own. Truth and reason are common to everyone, and no more belong to the man who first spoke them than to the man who says them later. It is no more according to Plato than according to me, since he and I understand and see it the same way. The bees plunder the flowers here and there, but afterward they make of them honey, which is all theirs; it is no longer thyme or marjoram. Even so with the pieces borrowed from others; he will transform and blend them to make a work of his own, to wit, his judgment. His education, work, and study aim only at forming this." (Montaigne, Essays).

Pedagogy comes from the greek Ï€Î±Î¹Î´Î±Î³Ï‰Î³Î¯Î± (paidagÅgeÅ), where Ï€Î±Î¹Î´ÏŒÏ‚ (país, genitive Ï€Î±Î¹Î´ÏŒÏ‚, paidos) means " child" and Î¬Î³Ï‰ (ágÅ) " lead"; hence one can understand pedagogy as " to lead the child". The slave who escorted the child to the school, but also carried his package and was responsible for his homework was the pedagogue.

In ancient Greek, paideia (Ï€Î±Î¹Î´ÎµÎ¯Î±) mean children education. In Athens children were taught grammar, rhetoric, mathematics, music, philosophy, geography, natural history and gymnastic. The paideia was supposed to elevate the child to transform him into an accomplished man.

One can define pedagogy as the method and practice of teaching, especially as an academic subject or theoretical concept (Oxford dictionary). Those methods and practices have considerably evolved over time and are still changing and evolving. Today, many challenging philosophies and ideas are competing. Debates over education are often heated, and always were (two centuries ago Rousseau's Emile: Or on Education was burned in Geneva).

Teaching and learning was always a domain of study and reflection. In every society and civilization, children are taught. For that purpose, institutions are created and philosophies of education developed. If some scholars consider the date 1657 as the birth of modern pedagogy, it is only because John Amos Comenius (28 March 1592 – 15 November 1670) (Murphy, 1995) teaching philosophy is very modern on many accounts. In a series of influential texts, Comenius developed the bedrock of pedagogy: 1. Education is for everybody without distinction of class, religion or sex. It is important to note that although at a time that considers women to be inferior to men, Comenius argue that girls have the same intellectual ability as boys; 2. Instruction should not be limited to intellectual and rational activities, but also must include manual training; 3. Learning practical subjects like geography, history or science rather than Latin (Comenius argued for less Latin, not to suppress it); 4. Giving importance to an artistic education, because art should be accessible to all; 5. Acquisition of knowledge should be pleasurable, not a chore. Comenius claims that teachers need to help the learners to build inner motivation. To this purpose they should use images. In his Orbis Sensualium Pictus (The Visible World in Pictures), notions are always accompanied by woodcuts. Naturally he opposed to any form of punishment, as the motivation for learning must come from the learner himself. In his Didactica magna Comenius promotes the creation of teacher training institutions. This philosophy was revolutionary at the time and its diffusion took many years.

The progressive education

One can trace back some of its principles to the Renaissance humanists, to Rabelais and Rousseau. In Emile: or On Education, Rousseau exposes his pedagogical project: Instruction should respect the nature of the child and not constraining his physical, intellectual and moral development; allowing him to fully develop in his natural self, but capable to live in society. It is still one of the most read and influential text on education. In Japan, it is a mandatory reading for primary teacher. However the birth of progressive education is at the end of the nineteenth century with the philosophy of John Dewey (1859-1952). The basic principles are in fact very close to Comenius's: 1. Motivation should come from the learner, rather than being externally stimulated; 2. Learning should be directly related to the learner; 3. Learners need positive learning experience (no frustration); 4. The learners must be active participant to the learning experience. His method is based on " hands-on learning", the teacher is a guide and the student actively learns by " doing". Like Rousseau, Dewey believed that education is not about acquiring a library of pre-determined skills, but rather realizing one's full potential. He was very critic to the traditional methods of instruction, where the learner passively receive the knowledge from the teacher, like an empty vase that would slowly be filled. Concomitently, he was very reserved on " child centered method". Although antagonist, both methods share the same flaw: they suppose a duality between the learner's experience and the knowledge taught (Westbrook, 2000). Although there were always many pedagogical philosophy in rupture with the traditional approach (e. g. Johann Heinrich Pestalozzi (1746-1827) founded schools in Switzerland after Rousseau's precepts), Dewey is seen as the " father" of all progressive education philosophies.

Another cornerstone in progressive education came with the work of Jean Piaget creator of the genetic epistemology (although the term is older and from James Mark Baldwin). The main idea is to see the process of acquiring knowledge as a construction. Piaget supposes the existence of deep psychological structure which regulate the learning process. Moreover after meeting Jean Dieudonné in Paris, Piaget would find direct relationship between Bourbaki's three mother structures (order structure, algebraic structure, topological structure) and the three structures he observed in children thinking (Behavioural schemata, Symbolic schemata, Operational schemata) (Wells, 2010). This identification would lead to the New Mathematics reform of the sixtees whose basis was that children's processes of acquiring (or constructing) mathematical concept were isomorphic to the way mathematics were constructed.

The traditional education

By traditional education, usually one refers to methods that are knowledge centered. The teacher is going to pass the science to the learner. The teacher would expose the knowledge and has the students practice through exercises. There is no differentiation and each learner treated the same. If one considers learning as making connections, traditional instruction supposes that all learners make the same connections. The school is conceived " as a place where certain information is to be given, where certain lessons are to be learned, or where certain habits are to be formed" (Dewey, 1897).

In fact the term traditional education is mostly used by practitioners of different pedagogical methods. When Dewey and his pupil William Heard Kilpatrick (1871-1965) defined progressive education it became necessary to make the distinction from former ways of teaching. So traditional education doesn't refer to any particular doctrine, but to what was thought to be common practice at that time. Actually one can draw many parallels between Dewey's progressive education and previous philosophies on education like Comenius's or even the Greek paideia. An essential difference between traditional and progressive pedagogies is to be found in the goals that they set and in the role that schools have. Traditional pedagogy aims are encapsulated by Montaigne's quote given above; it is for the learner to acquire knowledge and make it his own. On the other hand, progressive pedagogy is far more ambitious: "… meanwhile there are the enslaved human beings who must accomplish their own liberation. To develop their conscience and consciousness, to make them aware of what is going on, to prepare the precarious ground for the future alternatives – this is our task." (Marcuse, 1967). The goal set is infinitely higher and preferring a type of pedagogy is not anymore only a choice of the most effective practice. It becomes a political choice, which ramified to the role of schools in the society and how the society itself is organized.

English doesn't make a clear distinction between pedagogy and didactic. However many languages do. Didactic comes from the Greek didaktikos, meaning to ' teach'. If one considers the etymologies of the two words, it appears that pedagogy concerns the child or the learner while didactic would be about the teaching itself. Jean Houssaye defines a pedagogical triangle where the vertices are respectively knowledge, teacher and learner (Houssaye, 2000). Houssaye explains each of the side represent a process, the relation between two vertices. The side knowledge-teacher represents the didactic process, concerns with the knowledge, its organization and delivery. The teacher-learner side is about pedagogy, the relation between the teacher and the learner. The knowledge-learner side symbolizes the learning activity and the appropriation of knowledge. For Houssaye, traditional pedagogy only rests on the knowledge-teacher side while a full " child centered" approach would mostly rest on the teacher-learner side. A desirable pedagogical method would consider the whole triangle; the teacher would only purposely and temporarily emphasize one side of the triangle and would balance it at a later stage of the learning process.

Is there a mathematics pedagogy?

As previously stated one of the most important feature of mathematics is the language in which it is written. One of the difficulties in learning mathematics is the discrepancy between the formal definitions and the representations that one can have in his mind. If we consider an object as familiar as a straight line on a plane, what proper definition can we give of it? Mathematicians would define a straight line as an affine space of dimension 1. But using this definition suppose knowledge of linear algebra and that one reflected over the relationship between an affine space and a line drawn on a paper or board. A school definition could be an algebraic one: the set of points that satisfies some affine equation. With this example one can grasp the duality in doing mathematics. What one reads or writes is different from what is in ones mind. How does the teacher help the learner to create and connect a proper but personal set of images with the definitions?

One common method is to introduce new notions or concepts from concrete experience and problems. In doing so the teacher hopes to connect the abstract notions to concrete images and representations. Although sensible, this approach forgets a duality common to every mathematical notion; what Régine Douady calls the “ tool-object dialectic” (R. Douady, 1984). It is in fact a very simple idea as well as a very old one; but structuring all teaching in mathematics. It is illustrated in the opening of ‘ 2001: A Space Odyssey’ (Kubrick, 1968) when an ape uses a bone as tool and weapon. The scene shows the invention of the first tool and hence of the “ dialectic” relation between the object and the function assigned to it. Another illustration of this concept is seen in modern art where objects familiar to us are displayed in complete different setting (e. g. Juan Miró using discarded cans to conceive sculpture). Régine Douady contribution was to transpose this idea to mathematical objects. It is a common misconception to believe that mathematical objects are just pieces of a vast logical construct without finality or purpose. In fact mathematical objects are nothing if one forget their function, their tool side. It might have been that a particular concept or object have been created to solve a problem in mechanic or optic, or to solve a specific equation. Some of those objects were formed over many years, sometimes centuries. If they seem to have lost their original purpose and acquired some kind of transcendental existence it is only because unexpected usage have been found, they contributed solving unseen problems at the time of their conception. But also because, new objects acting on them have been created to solve increasingly harder problems. Hence, any number, function or shape carries this dual nature. They can be regarded in a structural way considering their relations with other similar objects, but they can be also studied on their own. After all many mathematicians started their journey contemplating the mysterious arrangements of decimals in a number, or observing the many striking features contain in a shape… This leads to a fundamental question for any teacher: given a concept, should it be introduced from its object or tool's aspect? Should children in primary school learn addition and division by 2 before being exposed to concrete sums or sharing's problems? How to introduce algebraic equations? Again through concretes problems or is it better to first gain familiarity with them? Is it possible for pupils to understand the concept of function through the study of one particular example?

None of those questions have an obvious answer. Whoever thinks the answer to be obvious should transfer the dilemma to other problems whose natures are not really different. Should we study the architecture of a bridge only through its function of crossing a river? Can we understand the beauty of a painting through the brushes of the painter? Régine Douady conclusions are very cautious: " From our experience, we can draw the following hypothesis; provided that ' enough' notions are introduced through the tool-object pedagogy, others can be directly provided by the teacher or the reading of a textbook. An important pedagogical problem is about the choices criterion, the organization and the articulation of the notions according to the way they are introduced. About this, we are not making any assumption, but giving examples of realization." (Douady, R, 1984, translation mine).

Conclusion

This essay focused on the didactical side of the teaching process explaining some of the challenges in teaching mathematics. Most of the properties and laws of mathematics, even when they seem obvious and clear have been developed over centuries (because none are obvious) and all carry implicit difficulties that the teacher is often unaware of. The tool/object dialectic is important to consider as it irrigates every mathematics pedagogy; from the abstract teaching of the sixties during the new math reform to the constructivist methods. However there is way out of the tool/object dialectic and as a conclusion I would