pgce mathematics

This problem involves fractions and the aim is to investigate how these numbers can be transformed to the next number in the sequence. How will I go about investigating this problem? First I will like to know where this sequence leads me. From this I will get a better idea of approaching the problem.

Approach

Using excel spreadsheet, starting with numerator and denominator being equal, ie a= 1 and b= 1. I found that the sequence of the transformation eventually converging to the square root of 2.

a equal to b (a= b= 1)

Sequence Sequence

Table 1 of of

a= b= 1 Numerator Denominator Result

1

1

1

3

2

1. 5

7

5

1. 4

17

12

1. 4166667

41

29

1. 41337931

99

70

1. 4142857

239

169

1. 4142012

577

408

1. 4142132

1393

985

1. 4142136

3363

2378

1. 4142136

From table 1, it was noticed that the sequence converges towards V2. I wanted to investigate what happens if a and b have different values and are not equal to each other. Again I used excel to develop the transformation. So my next step was to investigate what happens when a is greater than b.

Table 2

a= 2, b= 1

a greater than b by 1 (a> b)

Sequence Sequence

of of

Numerator Denominator Result

2

1

2

4

3

1. 3333333

10

7

1. 4285714

24

17

1. 4117647

58

41

1. 4146341

140

99

1. 4141414

Again I noticed the transformation converges to V2. To do a thorough investigation, I decided to use excel spreadsheet with a > b, by 2, 3, 4, and so on. It always gave the same result, transformation converging towards V2. This led me to my next step to investigate what happens when a is less than b.

Table3

a= 5, b= 7

a less than b by 2 (a

Sequence Sequence

of of

Numerator Denominator Result

5

7

0. 7142857

19

12

1. 5833333

43

31

1. 3870968

105

74

1. 4189189

253

432

1. 4143519

It is seen from table 3 that, even when a is less than b, it still converges to V2. On the excel spreadsheet I investigated various values of a and b, keeping a less than b. But it always gave me the same result, converging to V2. This is very interesting, could it to be anything to do with the coefficient of b. Since the coefficient of b is 2, and the transformation converges to V2. I will now investigate what happens if the coefficient of b to 3.

Investigating when the coefficient of b is changed to 3

a ï¿½ a + 3b , where a, b are whole numbers

b a + b

To investigate this phenomena of changing the coefficient of b to 3. I decided to use excel spreadsheet to see what number the sequence would converge to. My expectation was that it may converge to 3. The results which came from excel spreadsheet are shown in the below.

Table 4

a= b= 1 a equal to b (a= b)

Sequence Sequence

of of

Numerator Denominator Results

1

1

1

4

2

2

10

6

1. 6666667

28

16

1. 75

76

44

1. 7272727

208

120

17333333

568

328

1. 7317073

1552

896

1. 7321429

4240

2448

1. 7320574

11584

6688

1. 732049

As I suspected the result converges towards V3. Now my question is why does it converge to the square root of n? I am now in a stuck moment of how should I go about proving that it goes to Vn.

What I am now going to do is investigate the pattern being produced within the transformations of a and b. Hopefully this might help me to understand why it tends to Vn.

Investigating pattern of a and b in the formula a/b ï¿½ a + 2b

a + b

We are given the sequence

1 ï¿½ 3 ï¿½7ï¿½17ï¿½….

1 2 5 12

We now have to solve the next sequence of the algebra, Numerator and Denominator separately.

NUMERATOR

Adding the coefficients of a and b in the 2 previous terms, gives us the next term of the numerator.

1 + 2 = 3a

v v v

a a + 2b 3a + 4b

ï¿½ ï¿½ ï¿½

1 + 1 + 2 = 4b

This gives us the formula to find the numerator of the next term in the sequence, as shown below:

Un= 2Un-1 + Un-2

DENOMINATOR

Adding the coefficients of a and b in the 2 previous terms, gives us the next term in the denominator.

1 + 1 = 2a

v v v

b a + b 2a + 3b

ï¿½ ï¿½ ï¿½ ï¿½

1 + 1 + 1 = 3b

This gives us the formula to find the denominator of the next term in the sequence, as shown below:

Un = 2Un-1 + Un-2

Using the formula developed to find the next term in the sequence of a+2b a + b

1

1

3

2

7

5

17

12

41

29

99

70

239

169

577

408

1393

985

3363

2378

a

b

a+ 2b

a+b

3a + 4b

2a+3b

7a+10b

5a+7b

17a+24b

12a+17b

41a +58b

29a+41b

99a+140b

70a+99b

239a+338b

169a+239b

577a+816b

408a+577b

1393a+1970b

985a+1393b

OBSERVATION

By observation the sequence looked as if it is related Fibonacci sequences. I remember with the number cells you were given the first two terms, and then you added the two terms to give the next term. The sequence continued by keeps adding the last two terms to get the next term. But there is a difference with this sequence, because the last term is multiplied by 2. I did further research into the equation we derived earlier and came with PELL NUMBERS. This gave the following sequence:

1, 2, 5, 12, 29. 70, 169, 408….. and its equation was PK= 2PK-1 + PK-2, and its associated numbers are 1, 3, 17. 41, 99,….

This is the equation I had derived earlier. Our transformation also produced the sequences.

Now I am in stuck mode again, because I still haven't proved why the transformation tends to V2.

I have also noticed that the coefficient of b in the numerator is twice the coefficient of a in the denominator.

Also coefficient of a in the numerator is the same as coefficient of b in the denominator. Again, the coefficient of a in the denominator produce PELL numbers and whilst the coefficient of b produce its associated numbers. In the numerator only the coefficient of a produced the PELL numbers.

I will now investigate the pattern of a and b developed for the formula a+3b .

a+b

Investigating pattern of a and b in the formula a/b ï¿½ a + 3b

a + b

NUMERATOR

Adding the coefficients of a and b in the 2 previous terms, gives us the next term of the numerator.

1 + 3 = 4a

v v v

a a + 3b 4a + 6b

ï¿½ ï¿½ ï¿½ ï¿½

2*1 + 1 + 3 = 6b

This gives us the formula to find the numerator of the next term in the sequence, as shown below:

Un= 2Un-1 +2Un-2

DENOMINATOR

Adding the coefficients of a and b in the 2 previous terms, gives us the next term of the denominator.

1 + 1 = 2a

v v v

b a + b 2a + 4b

ï¿½ ï¿½ ï¿½ ï¿½

2*1 + 1 + 1 = 4b

This gives us the formula to find the denominator of the next term in the sequence, as shown below:

Un= 2Un-1 +2Un-2

Using the formula developed to find the next term in the sequence of a+3b

a + b

1

1

4

2

10

6

28

16

76

44

208

120

568

328

1552

896

4240

2448

11584

6688

a

b

a+ 3b

a+b

4a + 6b

2a+4b

10a+18b

6a+10b

28a+48b

16a+28b

76a +132b

44a+76b

208a+360b

120a+208b

568a+984b

328a+568b

1552a+2688b

896a+1552b

4240a+7344b

2448a+4240b

OBSERVATION

I have noticed a similar pattern recurring, the same as for formula (a+2b)/(a+b). The coefficient of b in the numerator is three times the coefficient of a in the denominator. As before it was two times greater.

Also coefficient of a in the numerator is the same as coefficient of b in the denominator. However, I cannot see any PELL number sequence in this transformation. So that theory has ‘ gone out of the window’. Now I am in a stuck moment. AHA!!! What kind of transformation converges to a certain number? Well it’s the Golden Ratio, which converges to 1. 6. Now I have to prove that our sequence converges to Vn.

GOLDEN RATIO GRAPH

2 ï¿½

1. 8

1. 6 ________________________ï¿½________________________

ï¿½

1. 4

1. 2

1 ï¿½ fib(i)__

fib(i-1)

0. 8

0. 6

0. 4

0. 2

0 2 4 6 8 10

Prove that a ï¿½ a+2b converges to V2.

b a+b

Dividing throughout by b

a/b ï¿½ a/b+2b/b

b/b a/b+b/b

a ï¿½ a/b+2

b a/b+1

Replace a/b with x

X = X+2

X+1

X(X+1)= X+2

X2 + X = X +2

X2 = 2

X = V2

Prove that a ï¿½ a+3b converges to V3

b a+b

Dividing throughout by b

a/b ï¿½ a/b+3b/b

b/b a/b+b/b

a ï¿½ a/b+3

b a/b+1

Replace a/b by X

X = X+3

X+1

X2 = 3X = V3

Prove that our transformation converges to Vn

a ï¿½ a+nb

b a+b

Dividing throughout by b

a/b ï¿½ a/b+nb/b

b/b a/b+b/b

a ï¿½ a/b+n

b a/b+1

Let X= a/b

X = X+n

X+1

X2+X = X+n

X2 = n

X = Vn

Prove transformation converges to Vn+1

a ï¿½ a+(n+1)b

b a+b

Dividing throughout by b

a/b ï¿½ a/b+(n+1)b/b

b/b a/b+b/b

a ï¿½ a/b+(n+1)

b a/b+1

Let X= a/b

X = X+(n+1)

X+1

X2+X = X+(n+1)

X2 = n+1

X = Vn+1 We have now proved that the transformation converges to Vn+1. But we need to know why does

it converge to Vn.

Why does our transformation converge to Vn

We have proved that X2= 2, this is the same as X*X= 2. Therefore the value of X= 2/X. However if we begin with a positive number X1, then either X1 or 2/X1 will be greater than V2 and the other will be smaller than V2. For example, if X1= 1 and 2/X1 = 2/1= 2, then X1 is less than V2 and 2 is greater than V2.

Hopefully using the average mean of X1 and 2/X1 will give us a better approximation to V2 than X1 does.

If given X1> 0, then to find the next term X2 in this particular sequence is:

n ? 1. Xn+1 = 1 (Xn + 2/Xn) for

2

Therefore, xn is converged to a particular value, then we have a limit of:

lim Xn+1 = lim (Xn/2 + 1/Xn)

Therefore, this property of limit, L must satisfy the condition L= L/2 + 1/L .

2L2= L2 + 2

2L2-L2= 2

L2= 2

From this we get L2= 2. If Xn > 0, then xn+1 will give the average of two positive numbers. Therefore, when X1> 0 this leads to positive limit, giving the positive square root of 2.

We need to show that the sequence has a limit, for positive initial prediction.

If x1 = 1, then using the following formula

Xn+1 = 1 (Xn + 2/Xn)……….(1)

2

So the next term of this sequence is:

X2 = 1 (1 + 2/1)

2

X2 = 3 Putting the value of X2 into the above equation 1, we get the next term of X3 = 17 and so on.

2 12

However we notice the initial term of X1= 1 is less than V2, the next term X2= 3/2 is greater than V2. From this step, the sequence starts to decrease and is bounded below V2. Therefore the Monotone Convergence Theorem implicates the existing of the limit.

So we have proved that if X1 > 0, then at X2 it starts to monotone decrease and is bounded below by V2.

This means the transformation converges.