Example of research paper on fundamental theorem of algebra

Law, Evidence

The fundamental theorem of algebra (FToA) is states that each non-constant single-variable polynomial with complex coefficients has at least one complex root. In fact, there are many equivalent formulations, such as this: every real polynomial can be represented as a product of real linear and real quadratic factors.

Early studies equations by al-Khwarizmi (from 800) are devoted only to positive real roots and have no relation to the FToA. Cardano was the first to realize that it is possible to work with the values more general than real numbers (Geuvers, Wiedijk & Zwanenburg, 2002). This discovery was made in the study of the roots of a cubic equation formula. This formula when applied to the equation x3= 15x+4 gives the answer, in which there is -121, although Cardano knew that x = 4 is a root of the equation. He could use his " complex numbers'' to get the right answer, but he still could not explain his mathematics.

Bombelli in " Algebra," published in 1572, gave a proper set of rules of action with these " complex numbers.'' Descartes in 1637 said that it was possible to “ assume” for every equation of n-th degree n roots, but these roots do not match any real values. Wyeth cited equation of n degree with n roots, but the first to claim that there is always n solutions, was Dutch mathematician Albert Girard in 1629. However, he did not prove that the solutions have the form a+bi, where a and b are real numbers, so he allowed for the possibility that the solutions belong to the field, comprising C. In fact, it was a problem of the FToA for many years while mathematics accepted Albert Girard’s statement as self-evident.

Harriot knew that a polynomial that turns to 0 at t, has a root x = t, but it was not widely known before the statement of Descartes in 1637, so Albert Girard did not bring sufficient justification for an accurate understanding of the problem. The “ proof” of FToA incorrectness was given by Leibniz in 1702, when he claimed that the polynomial x4+t4 cannot be written as the product of two real quadratic factors. His mistake stemmed from a misunderstanding that i can be represented in the form a+bi, where a and b are real.

D'Alembert in 1746 made the first serious attempt to prove the FToA. For the polynomial f he chooses real b and c such that fb= c. He shows that there are complex numbers z1 and w1 such that:

z1 Then he presents an iterative process that converges to the root of f. His proof has several drawbacks. First, it uses without proof a lemma, which was proved in 1851 by Puiseux, but this proof uses the FToA. Secondly, he did not have the necessary knowledge to use compactness to give the final proof of convergence. Despite this, the ideas of the proof are important (Derksen, 2003).

Euler was soon able to prove that every real polynomial of degree n (n≤6) has exactly n complex roots. In 1749, he tried to prove it in the general case, i. e. he tried to prove the FToA for real polynomials: every polynomial of n-th degree with real coefficients has exactly n zeroes in C.

In 1814, the Swiss accountant Jean Robert Argand published FToA proof, which is probably the easiest of all the evidence. It is based on the idea of d'Alembert of 1746. Argand simplifies the idea of d'Alembert to use the general theorem on the existence of a minimum of a continuous function (Richman, 2000). This proof was not strict, since the lower limit of the general concept has been developed at the time. Proof of Argand became famous when it was given in Chrystal's algebra textbook in 1886.

In two years after the proof of Argand, in 1816, Gauss published the second proof of the FToA. Gauss used Euler approach, but instead of working with roots that cannot exist, Gauss works with unknowns. This proof is complete and correct. Third proof of Gauss (also 1816) is, like the first, of topological character. Gauss in 1831 introduced the concept of " complex number".

Of course, the evidence described above are valid only in the existence of the modern result about existence of the expansion field of any polynomial. Frobenius in Basel at the celebrations dedicated to the bicentenary of the birth of Euler said that Euler gave the most algebraic proof of the existence of roots of the equation based on the assumption that every real equation of odd degree has a real root. He believed that it was unfair to attribute this proof solely to Gauss, which only added the finishing touches.

Argand's proof is the only proof of existence, and it in no way allows us to find the roots. In 1859, Weierstrass attempted constructive proof, but only in 1940, the constructive version of the proof was given by Hellmuth Kneser. This proof was further simplified in 1981 by Martin Kneser, son of Helmut Kneser.

There are several conclusions of the FToA. Any polynomial pz= anzn+an-1zn-1++a1z+ao of the degree n> 1 with complex coefficients a0, a1,, an-1, an≠0 can be represented as a product of linear binomials:

anzn+an-1zn-1++a1z+ao= anz-z1k1z-z2k2••z-zsks,

where z1, z2, , zs – are the roots of polynomial function of the degree k1, k2, , ks respectively, and k1+k2++ks= n. In other words, a polynomial of n-th degree has exactly n roots, if every root is considered as many times as its multiplicity (Milewski, 2001).

Conclusion 2. If the polynomials p(z) and q(z), degrees of which are not higher than n, have the same value at more than n different values of z, then these polynomials are equal: p(z) = q(z).

In fact, by conditions, polynomial pz-q(z) has more than n roots, although its degree is less than or equal to n, which contradicts conclusion 1 of the FToA. Hence, it is a polynomial of degree zero p(z)-q(z) = a0. Because it has roots, a0= 0. Consequently, p(z)-q(z) = 0, i. e. p(z)= q(z).

This conclusion allows us to consider the polynomial p(x) as a function of the variable x, since the equality of polynomials p(x) = q(z), defined above as the equality of the coefficients at the same degrees of x, coincides with the notion of equality p(x) = q(x) of two functions for all values of x.

Conclusion 3. If the complex (but not real) number c is a root of polynomial p(c) with real coefficients, then the conjugate c is a root of the same multiplicity. In fact, if c is a root of multiplicity k, then for it the following conditions are valid:

pc= 0, p'c= 0, , pk-1c= 0, pk(c)≠0.

of the conditions

pc= p(c)= 0, p'c= p'(c)= 0, , pk-1c= pk-1c= 0, pkc= pk(c)≠0

It follows that c is a root of the same degree k.

Conclusion 4. Every polynomial px= anxn+an-1xn-1++a1x+ao with real coefficients is represented as a product of linear binomials and quadratic trinomials (with negative discriminant):

anxn+an-1xn-1++a1x+ao= anx-x1k1x-x2k2••xxsks∙x2+p1x+q1m1∙x2+p2x+q2m2∙∙x2+prx+qrmr,

where x1, x2, , xs are the real roots of degrees k1, k2, , ks, and k1+k2+ +ks+2m1+2m2++2mr= n.

Conclusion 5. Odd degree polynomial with real coefficients always has at least one real root.

Even degree polynomial with real coefficients cannot have real roots (in this case there are no linear binomials x-x1, x-x2, , (x-xs).

References

Derksen, H. (2003). The fundamental theorem of algebra and linear algebra. The American mathematical monthly, 110(7), 620-623.

Geuvers, H., Wiedijk, F., & Zwanenburg, J. (2002). A constructive proof of the fundamental theorem of algebra without using the rationals. In Types for Proofs and Programs (pp. 96-111). Springer Berlin Heidelberg.

Milewski, R. (2001). Fundamental theorem of algebra. Formalized Mathematics, 9(3), 461-470.

Richman, F. (2000). The fundamental theorem of algebra: a constructive development without choice. Pacific J. Math, 196(1), 213-230.