Rigidity of complete minimal submanifolds in spheres

1. Introduction

Let $(x , u (x))$ be a minimal graph in $\mathbb{R}^2 \times \mathbb{R}$, which means that $u (x)$ solves the equation

$$\operatorname{div} \left(\frac{\nabla u}{1 + | \nabla u |^2} \right) = 0 .$$

The celebrated Bernstein theorem states that the complete minimal graphs in $\mathbb{R}^3$ are planes. The works of Fleming [9], Almgren [1], and Neto and Wang [16] tell us that the Bernstein theorem is valid for complete minimal graphs in $\mathbb{R}^{n+1}$ provided that $n \leq 7$. Counterexamples to the theorem for $n \geq 8$ have been found by Bombieri et al.[2] and, later, by Lawson [13]. On the other hand, do Carmo and Peng [6] and Fischer-Colbrie and Schoen [10] proved independently that a completely stable minimal surface in $\mathbb{R}^3$ must be a plane, a result that generalizes the Bernstein theorem. For the high-dimensional case, it is an open question whether the completely oriented stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ (for $3 \leq n \leq 7$) are hyperplanes. However, it has been proved by do Carmo and Peng [6] that a complete stable minimal hypersurface M in $\mathbb{R}^{n+1}$ is a hyperplane if

$$\lim_{R \to \infty} \frac{\int_{B_{x_0}(R)} | h |^2 \, dv}{R^{2q+2}} = 0 , \quad q < 2n ,$$

where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$. Many interesting generalizations of the do Carmo-Peng theorem have been obtained (see, e. g., [7 , 15 , 16 , 18]). By definition, the hyperbolic space $\mathbb{H}^{n+p}$ is a Riemannian manifold with sectional curvature -1 which is simply connected, complete, and $(n + p)$-dimensional. In hyperbolic space, some

results similar to the do Carmo-Peng theorem have been derived. Xia and Wang [20] studied complete minimal submanifolds in a hyperbolic space and obtained the following result.

T HEOREM 1. 1. [20] *For n ≥ 5, let M be an n - dimensional complete immersed minimal submanifold in a hyperbolic space* $\mathbb{H}^{n+p}$ *, and let h be the second fundamental form of M . Assume that*

$$\lim_{R \to \infty} \sup \int_{B_{x_0}(R)} |h|^2 \, dv \, R^2 = 0 . \quad (1.1)$$

If there exists a positive constant C depending only on n and p such that

$$\int_M |h|^n \, dv < C,$$

then M is totally geodesic .

Recently, de Oliveira and Xia [8] improved Theorem 1. 1 as follows.

T HEOREM 1. 2. [8] *For n ≥ 4, let M be an n - dimensional complete immersed minimal submanifold in a hyperbolic space* $\mathbb{H}^{n+p}$ *such that n and p satisfy* $(n^2 - 6n + 1) + 8p > 0$ *. Assume that*

$$\lim_{R \to \infty} \sup \int_{B_{x_0}(R)} |h|^d \, dv \, R^2 = 0 ,$$

where d is a constant with the following properties :

(1) *if p = 1 and n ≥ 4, then*

$$d \in \left(\frac{n-1}{n}, \frac{(n-2)(n-1)}{n} \right) ;$$

(2) *if p > 1 and n > 5, then*

$d \in (n-1)^2 2n \left(1 - \sqrt{1 - \frac{4(n-1)}{2(n-2p)}}, 1 + \sqrt{1 - \frac{4(n-1)}{2(n-2p)}} \right)$.

Then there exists a positive constant C depending only on n , p , and d such that M is totally geodesic if

$\int_M |h|^n \, dv < C$.

The unit sphere $????^{n+p}$ is a Riemannian manifold with sectional curvature 1 which is simply connected, complete, and ($n + p$)-dimensional. Many results are available on the classification of compact minimal submanifolds in the unit sphere. Simons [17] calculated the Laplacian of $|h|^2$ of minimal submanifolds in a space form. As a consequence of Simons' formula, if M is a compact minimal submanifold in $????^{n+p}$ and $|h|^2 \leq \frac{np}{2p-1}$, then either M is totally geodesic or $|h|^2 = \frac{np}{2p-1}$. In the latter case, Chern et al.[3] further proved that M is either a Clifford hypersurface or a Veronese surface in $????^4$. Li and Li [14] and Chen and Xu [4] proved independently that M is either a totally geodesic submanifold or a Veronese surface in $????^4$ if $|h|^2 \leq \frac{2}{3}n$ everywhere on M . This result improves the pinching constant in Simons' formula. Deshmukh [5] studied n -dimensional compact minimal submanifolds in $????^{n+p}$ with scalar curvature S satisfying the pinching condition $S > n(n - 2)$ and proved that for $p \leq 2$ these submanifolds are totally geodesic.

The above results are rigidity theorems valid in the unit sphere, which characterize the behavior of minimal submanifolds. In this paper, we use the methods of minimal submanifolds in Euclidean space and hyperbolic space

to investigate the rigidity of complete minimal submanifolds in spherical space. The main theorems are as follows.

THEOREM 1. 3. *For $n \geq 3$, let M be an n - dimensional complete minimal submanifold in the unit sphere* ???? $^{n+p}$. *We further assume that (1. 1) holds. If*

$$\int M |h|^n dv < Cn(n,p)$$

with $C(n,p) = (c(n))^{-1}(n^2+7)^{-\frac{1}{2}}(2b(p))^{-1} + (npb(p))^{-1}$, *where* $c(n) = 2n(1+n)^{1+\frac{1}{n}}(n-1)^{-1}\omega_{n-1}^n$, ω_n *is the volume of the unit ball in* $\mathbb{R}^n$, $b(1) = 1$, *and* $b(p) = \frac{3}{2}$ *if* $p > 1$, *then M is totally geodesic*.

In [20], Xia and Wang believed that the condition (1. 1) is not necessary. It is therefore interesting to see whether we can remove condition (1. 1) from Theorem 1. 3. In this case, we get a positive answer.

THEOREM 1. 4. *For $n \geq 3$, let M be an n - dimensional complete minimal submanifold in the unit sphere* ???? $^{n+p}$. *If*

$$\int M |h|^n dv < \tilde{C}n(n,p)$$

with $\tilde{C}(n,p) = (nc(n))^{-1}\frac{2(n-1)}{(n^2+7)b(p) + 4n(n^2+7)pb(p)}$, *where* $c(n) = 2n(1+n)^{1+\frac{1}{n}}(n-1)^{-1}\omega_{n-1}^n$, ω_n *is the volume of the unit ball in* $\mathbb{R}^n$, $b(1) = 1$, *and* $b(p) = \frac{3}{2}$ *if* $p > 1$, *then M is totally geodesic*.

Remark 1. 5. By using Simons' formula and the technique developed in do Carmo and Peng's paper, we obtain Theorem 1. 4. The constant $\tilde{C}(n, p)$ in Theorem 1. 4 is smaller than $C(n, p)$ in Theorem 1. 3.

We also investigate stable minimal hypersurfaces in the unit sphere and obtain a result similar to do Carmo and Peng's theorem. A minimal hypersurface M in a Riemannian manifold N is said to be stable if for each $f \in C_0^\infty(M)$,

$$\int_M (|\nabla f|^2 - (|h|^2 + \bar{Ric}(\nu, \nu)) f^2) \, dv \geq 0, \quad (1.2)$$

where $\bar{Ric}$ is the Ricci curvature of N and ν is the unit normal vector of M.

T HEOREM 1. 6. *For $n \geq 2$, let M be an n - dimensional complete stable minimal hypersurface in the unit sphere* $????^{n+1}$ *. If*

$$\lim_{R \to \infty} \sup \int_{B_{x_0}(R)} |h|^{2\delta} dv \, R^2 = 0, \quad 1 - \frac{2}{n} < \delta < 1 + \frac{2}{n}, \quad (1.3)$$

where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$, then M is totally geodesic .

2. Preliminaries

Let M be an n -dimensional complete submanifold in the $(n + p)$ -dimensional unit sphere $????^{n+p}$. We will use the following convention on the range of indices unless specified otherwise:

$$1 \leq A, B, C, \ldots \leq n + p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$

We choose a local field of orthonormal frame $\{ e_1, e_2, ..., e_{n+p} \}$ in ???? ^{n} $^{+\,p}$ such that, restricted to M, $\{ e_1, e_2, ..., e_n \}$ is tangent to M and $\{ e_{n+1}, ..., e_{n+p} \}$ normal to M. Let $\{\omega_A\}$ be the field of dual frame and $\{\omega_{AB}\}$ the connection 1-form of ???? $^{n+p}$. Restricting these forms to M, we have

$$\omega_{i\alpha} = \sum_j h_{ij\alpha} \omega_j, \quad h = \sum_{i,j,\alpha} h_{ij\alpha} \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{i,\alpha} h_{ii\alpha} e_\alpha,$$

where h and ξ are the second fundamental form and the mean curvature vector of M, respectively. We define

$$H = |\xi|, \quad |h|^2 = \sum_{i,j,\alpha} (h_{ij\alpha})^2, \quad |\nabla h|^2 = \sum_{i,j,k,\alpha} (h_{ijk\alpha})^2,$$

where $h_{ijk\alpha}$ is the component of the covariant derivative of $h_{ij\alpha}$. When M is minimal, we obtain the Simons' formula [3 , 17]

$$\frac{1}{2} \Delta |h|^2 = |\nabla h|^2 + n|h|^2 - \sum_{i,j,k,l,\alpha,\beta} h_{ij\alpha} h_{ij\beta} h_{kl\alpha} h_{kl\beta} - \sum_{i,j,\alpha,\beta} \left(\sum_k (h_{ik\alpha} h_{kj\beta} - h_{jk\alpha} h_{ki\beta}) \right)^2 . \quad (2.1)$$

The last terms in (2. 1) can be estimated as [14]

$$-\sum_{i,j,k,l,\alpha,\beta} h_{ij\alpha} h_{ij\beta} h_{kl\alpha} h_{kl\beta} - \sum_{i,j,\alpha,\beta} \left(\sum_k (h_{ik\alpha} h_{kj\beta} - h_{jk\alpha} h_{ki\beta}) \right)^2 \geq - b(p)|h|^4, \quad (2.2)$$

with $b(1) = 1$ and $b(p) = \frac{3}{2}$ if $p > 1$. We need the following estimate:

L EMMA 2. 1. [19] *Let M be an n - dimensional immersed submanifold with parallel mean curvature in the space form* $M^{n+p}(k)$. *Then*

$$|\nabla h|^2 - |\nabla |h||^2 \geq \frac{2}{np} |\nabla |h||^2 .$$

We also need the following Hoffman-Spruck Sobolev inequality.

LEMMA 2. 2. [12] *Let M be an n - dimensional complete submanifold in a Hadamard manifold and let* $\psi \in C_0^1(M)$ *. Then*

$$\left(\int_M \psi^{\frac{n}{n-1}} dv \right)^{\frac{n-1}{n}} \leq c(n) \int_M (|\nabla \psi| + n|H|\psi) dv,$$

where $c(n) = 2^n (1+n)^{1+\frac{1}{n}} (n-1)^{-1} \omega_{n-1}^{\frac{1}{n}}$ *and* ω_n *is the volume of the unit ball in* $\mathbb{R}^n$ *.*

From Lemma 2. 2, we have the following estimate.

LEMMA 2. 3. [11] *For* $n \geq 3$, *let M be an n - dimensional complete minimal submanifold in* ???? $^{n+p}$ *and let* $\psi \in C_0^1(M)$ *. Then*

$$\left(\int_M \psi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq 2(n^2+7) c^2(n) \int_M (|\nabla \psi|^2 + |\psi|^2) dv.$$

3. Proofs of the Main Theorems

PROOF OF THEOREM 1. 3: Noting that

$$\frac{1}{2}\Delta|h|^2 = |\nabla|h||^2 + |h|\Delta|h|,$$

it follows from (2. 1) and (2. 2) that

$$|\nabla|h||^2 + |h|\Delta|h| \geq |\nabla h|^2 + n|h|^2 - b(p)|h|^4.$$

From Lemma 2. 1, we have

$$|h|\Delta|h| \geq \frac{2}{np}|\nabla|h||^2 + n|h|^2 - b(p)|h|^4. \quad (3. 1)$$

Given $\eta \in C_0^\infty(M)$, multiplying (3.1) by η^2 and integrating over M gives

$$\int_M \eta^2 |h| \Delta |h| \, dv + \int_M b(p) \eta^2 |h|^4 \, dv \geq 2np \int_M |\nabla |h||^2 \eta^2 \, dv + n \int_M |h|^2 \eta^2 \, dv, \quad (3.2)$$

which implies

$$-\int_M 2\eta |h| \langle \nabla |h|, \nabla \eta \rangle \, dv + \int_M b(p) \eta^2 |h|^4 \, dv \geq (1 + 2np) \int_M |\nabla |h||^2 \eta^2 \, dv + n \int_M |h|^2 \eta^2 \, dv. \quad (3.3)$$

Further, applying Hölder's inequality and taking $\psi = |h|\eta$ in Lemma 2.3, one verifies that

$$\int_M \eta^2 |h|^4 \, dv \leq \left(\int_M |h|^n \, dv \right)^{\frac{2}{n}} \left(\int_M (\eta |h|)^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq 2(n^2 + 7) c^2(n) \left(\int_M |h|^n \, dv \right)^{\frac{2}{n}} \int_M (|\nabla(\eta |h|)|^2 + |h|^2 \eta^2) \, dv. \quad (3.4)$$

Setting

$$l = 2b(p)(n^2 + 7) c^2(n) \left(\int_M |h|^n \, dv \right)^{\frac{2}{n}},$$

from (3.3) and (3.5) we may estimate

$$l \int_M |\nabla \eta|^2 |h|^2 \, dv + (l - 1) \int_M 2\eta |h| \langle \nabla |h|, \nabla \eta \rangle \, dv \geq (1 + 2np - l) \int_M |\nabla |h||^2 \eta^2 \, dv + (n - l) \int_M |h|^2 \eta^2 \, dv. \quad (3.5)$$

By assumption,

$$\left(\int_M |h|^n \, dv \right)^{\frac{1}{n}} < c^{-1}(n)(n^2 + 7)^{-\frac{1}{2}} (2b(p))^{-1} + (npb(p))^{-1},$$

and it is easy to see that

$$1 + 2np - l > 0.$$

Therefore, we can find a $\theta > 0$ such that

$$1 + 2np - l \geq \theta.$$

On the other hand, for any $\varepsilon > 0$ we have

$$(l-1) \int_M 2\eta |h| \langle \nabla |h|, \nabla \eta \rangle \, dv \leq |l-1| \varepsilon \int_M \eta^2 |\nabla |h||^2 \, dv + |l-1| \varepsilon^{-1} \int_M |\nabla \eta|^2 |h|^2 \, dv. \quad (3.6)$$

Thus, when $|l-1|\varepsilon \leq \frac{\theta}{2}$, we obtain

$$(l + |l-1|\varepsilon^{-1}) \int_M |\nabla \eta|^2 |h|^2 \, dv \geq \frac{\theta}{2} \int_M |\nabla |h||^2 \eta^2 \, dv + (n-l) \int_M |h|^2 \eta^2 \, dv. \quad (3.7)$$

Fix a point $x_0 \in M$ and choose $\eta \in C_0^\infty(M)$ as

$$\eta = \{ 1 \text{ on } B_{x_0}(R), \, 0 \text{ on } M B_{x_0}(2R), \, |\nabla \eta| \leq \frac{1}{R} \text{ on } B_{x_0}(2R) B_{x_0}(R), \quad (3.8)$$

with $0 \leq \eta \leq 1$, where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$. Substituting the above η into (3.7) and letting $R \to \infty$, we deduce that

$$\int_M |\nabla |h||^2 \, dv \leq 0, \quad \int_M |h|^2 \, dv \leq 0.$$

Hence $|h|^2 = 0$, that is, M^n is totally geodesic. □

Proof of Theorem 1.4: Direct computation yields

$$\Delta |h|^\delta = \delta(\delta-1)|h|^{\delta-2}|\nabla|h||^2 + \delta|h|^{\delta-1}\Delta|h|. \quad (3.9)$$

Multiplying (3. 9) by $|h|^\delta$ and using (3. 1), we infer that

$$|h|^\delta \Delta|h|^\delta = \delta - 1\delta|\nabla|h|^\delta|^2 + \delta|h|^{2\delta-2}|h|\Delta|h| \geq \delta - 1\delta|\nabla|h|^\delta|^2 + \frac{2\delta}{np}|\nabla|h||^2|h|^{2\delta-2} + n\delta|h|^{2\delta} - \delta b(p)|h|^{2\delta+2} = \left(\delta - 1\delta + \frac{2}{np\delta}\right)|\nabla|h|^\delta|^2 + (n\delta - b(p)\delta|h|^2)|h|^{2\delta}. \quad (3.10)$$

Let $\eta \in C_0^\infty(M)$. Multiplying (3. 10) by η^2 and integrating over M yields

$$\int_M \eta^2 |h|^\delta \Delta|h|^\delta dv \geq \left(\delta-1\delta + \frac{2}{np\delta}\right)\int_M \eta^2|\nabla|h|^\delta|^2 dv + n\delta\int_M \eta^2|h|^{2\delta}dv - b(p)\delta\int_M \eta^2|h|^{2\delta+2}dv. \quad (3.11)$$

It follows from the divergence theorem and (3. 11) that

$$b(p)\delta\int_M \eta^2|h|^{2\delta+2}dv - \int_M 2\eta|h|^\delta\langle\nabla\eta,\nabla|h|^\delta\rangle dv \geq \left(2 + \frac{2-np}{np\delta}\right)\int_M \eta^2|\nabla|h|^\delta|^2 dv + n\delta\int_M \eta^2|h|^{2\delta}dv. \quad (3.12)$$

Applying Hölder's inequality and taking $\psi = |h|^\delta\eta$ in Lemma 2. 3, we have

$$\int_M \eta^2|h|^{2\delta+2}dv \leq \left(\int_M |h|^n dv\right)^{\frac{2}{n}}\left(\int_M (\eta|h|^\delta)^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \frac{2n^2 c^2(n)}{n}\left(\int_M |h|^n dv\right)^{\frac{2}{n}}\int_M \left(|\nabla(\eta|h|^\delta)|^2 + |h|^{2\delta}\eta^2\right)dv. \quad (3.13)$$

Substituting (3. 13) into (3. 12) yields

$$\delta l\int_M |h|^{2\delta}|\nabla\eta|^2 dv + (\delta l - 1)\int_M 2\eta|h|^\delta\langle\nabla\eta,\nabla|h|^\delta\rangle dv \geq \left(2 + \frac{2-np}{np\delta} - l\delta\right)\int_M \eta^2|\nabla|h|^\delta|^2 dv + (n\delta - l\delta)\int_M \eta^2|h|^{2\delta}dv, \quad (3.14)$$

where $l = 2b(p)(n^2+7)c^2(n)(\int_M |h|^n)^{\frac{2}{n}}$. Further, using the Cauchy-Schwarz inequality, for each $\varepsilon > 0$ we obtain

$$(\delta l - 1)\int_M 2\eta|h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle dv \le |\delta l - 1|\varepsilon \int_M \eta^2 |\nabla|h|^\delta|^2 dv + |\delta l - 1|\varepsilon^{-1} \int_M |h|^{2\delta}|\nabla\eta|^2 dv. \quad (3.15)$$

Therefore

$$(\delta l + |\delta l - 1|\varepsilon^{-1})\int_M |h|^{2\delta}|\nabla\eta|^2 dv \ge (2 + \frac{2-n}{pn}p\delta - l\delta - |\delta l - 1|\varepsilon)\int_M \eta^2 |\nabla|h|^\delta|^2 dv + (n\delta - l\delta)\int_M \eta^2 |h|^{2\delta} dv. \quad (3.16)$$

By the assumption in the theorem that

$$(nc(n))^{-1}\frac{2(n-1)}{(n^2+7)b(p)+4n(n^2+7)pb(p)} > (\int_M |h|^n dv)^{\frac{1}{n}},$$

we have

$$2 + \frac{2-n}{pn}p\delta - l\delta > 0.$$

Choosing ε sufficiently small, we can get

$$2 + \frac{2-n}{pn}p\delta - l\delta - |\delta l - 1|\varepsilon > 0.$$

Defining the cut-off function as in (3.8) and taking $\delta = \frac{n}{2}$ in (3.16), we obtain

$$(\frac{n}{2}l + |\frac{n}{2}l - 1|\varepsilon^{-1})\int_M |h|^n dv_{R^2} \ge (\frac{n}{2}l + |\frac{n}{2}l - 1|\varepsilon^{-1})\int_{B_{x_0}(2R)} |h|^n dv_{R^2} \ge (2 + \frac{4-2n}{pn^2}p - \frac{n}{2}l - |\frac{n}{2}l - 1|\varepsilon)\int_{B_{x_0}(R)} |\nabla|h|^{\frac{n}{2}}|^2 dv + (\frac{1}{2}n^2 - \frac{n}{2}l)\int_{B_{x_0}(R)} |h|^n dv. \quad (3.17)$$

Since

$$\left(\int M \, |h|^n \, dv \right)^{\frac{1}{n}} < \tilde{C}(n, p),$$

upon taking $R \to \infty$ we have

$$\int M \, |h|^n \, dv \, R^2 \to 0.$$

This and (3. 17) imply $\nabla|h| = 0$ and $|h| = 0$, that is, M^n is totally geodesic.

□

Proof of Theorem 1. 6: Since M is a stable minimal hypersurface in the unit sphere ???? $^{n+1}$, (1. 2) holds on M. Let $\eta \in C_0^\infty(M)$. Replacing f by $\eta|h|^\delta$ in (1. 2) and taking $\overline{\mathrm{Ric}}(\nu, \nu) = n$ give

$$\int M \, |\nabla(\eta|h|^\delta)|^2 \, dv \geq \int M \, \eta^2 |h|^{2\delta+2} \, dv + n \int M \, \eta^2 |h|^{2\delta} \, dv,$$

that is,

$$\int M \, |\nabla\eta|^2 |h|^{2\delta} \, dv + \int M \, \eta^2 |\nabla|h|^\delta|^2 \, dv + 2 \int M \, \eta |h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle \, dv - n \int M \, \eta^2 |h|^{2\delta} \, dv \geq \int M \, \eta^2 |h|^{2\delta+2} \, dv. \quad (3. 18)$$

Substituting (3. 18) into (3. 12) and noting that $b(1) = 1$, we obtain

$$\delta \int M \, |\nabla\eta|^2 |h|^{2\delta} \, dv + 2(\delta-1) \int M \, \eta |h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle \, dv \geq (2 + 2 - nn\delta - \delta) \int M \, \eta^2 |\nabla|h|^\delta|^2 \, dv + 2n\delta \int M \, \eta^2 |h|^{2\delta} \, dv. \quad (3. 19)$$

Using $1 - \frac{2}{n} < \delta < 1 + \frac{2}{n}$, we see that

$$2 + 2 - nn\delta - \delta > 0.$$

Further, for any $\varepsilon > 0$, it follows from the Cauchy-Schwarz inequality that

$$\varepsilon |\delta - 1| \int_M \eta^2 |\nabla|h|^\delta|^2 \, dv + \varepsilon^{-1} |\delta - 1| \int_M |\nabla \eta|^2 |h|^{2\delta} \, dv \geq 2 (\delta - 1) \int_M \eta |h|^\delta \langle \nabla \eta, \nabla |h|^\delta \rangle \, dv. \quad (3.20)$$

Combining (3.20) and (3.19) gives

$$(\delta + \varepsilon^{-1} |\delta - 1|) \int_M |\nabla \eta|^2 |h|^{2\delta} \, dv \geq \left(2 + \frac{2-n}{n} \delta - \delta - \varepsilon |\delta - 1|\right) \int_M \eta^2 |\nabla|h|^\delta|^2 \, dv + 2n\delta \int_M \eta^2 |h|^{2\delta} \, dv. \quad (3.21)$$

Choosing ε sufficiently small, we can obtain

$$2 + \frac{2-n}{n} \delta - \delta - \varepsilon |\delta - 1| > 0.$$

Furthermore, defining the cut-off function as in (3.8) and using the assumption (1.3) yield $\nabla|h| = 0$ and $|h| = 0$, that is, M^n is totally geodesic. $\square$

4. Conclusion

In this paper, by using Simons' formula, a Sobolev-type inequality as in Chen and Xu [4], and the technique of do Carmo and Peng, we obtain rigidity theorems for minimal submanifolds in ????$^{n+p}$. Compared with Theorem 1. 1, Theorem 1. 4 removes the condition on the growth of the norm of the second fundamental form. Moreover, our results require only $n \geq 3$, whereas Theorems 1. 1 and 1. 2 require $n \geq 5$ and $n \geq 4$, respectively. Whether the pinching constant for the total curvature in Theorem 1. 4 is optimal remains an open question and is a topic of future research.

Data Availability Statement

The raw data supporting the conclusions of this article will be made available

by the authors, without undue reservation.

Author Contributions

The author confirms being the sole contributor of this work and has approved

it for publication.

Funding

Conflict of Interest

The author declares that the research was conducted in the absence of any

commercial or financial relationships that could be construed as a potential

conflict of interest.

Acknowledgments

References

1. Almgren F. Some interior regularity theorems for minimal surfaces and an

extension of Bernstein's theorem. *Ann Math.* (1966)84: 277–92. doi: 10.

2307/1970520

CrossRef Full Text | Google Scholar

2. Bombieri E, De Giorgi E, Giusti E. Minimal cones and the Bernstein problem. *Invent Math.* (1969)7: 243–68. doi: 10. 1007/BF01404309

CrossRef Full Text | Google Scholar

3. Chern S, do Carmo M, Kobayashi S. Minimal submanifolds of a sphere with second fundamental form of constant length. In: Tenenblat K, editor. *Manfredo P. do Carmo – Selected Papers* . Berlin; Heidelberg: Springer (1970). p. 59–75. doi: 10. 1007/978-3-642-25588-5_5

CrossRef Full Text | Google Scholar

4. Chen Q, Xu S. Rigidity of compact minimal submanifolds in a unit sphere. *Geome Dedicata* . (1993)45: 83–8. doi: 10. 1007/BF01667404

CrossRef Full Text | Google Scholar

5. Deshmukh S. Rigidity of compact minimal submanifolds in a sphere. *Pacific J Math* . (2000)193: 31–44. doi: 10. 2140/pjm. 2000. 193. 31

CrossRef Full Text | Google Scholar

6. do Carmo M, Peng C. Stable complete minimal hypersurfaces. *Proc Beijing Symp Diff Equat Diff Geometry* . (1980)3: 1349–58.

7. do Carmo M, Zhou D. Bernstein-type theorems in hypersurfaces with constant mean curvature. *An Acad Bras Ciênc* . (2000)72: 301–10. doi: 10. 1590/S0001-37652000000300003

PubMed Abstract | CrossRef Full Text | Google Scholar

8. de Oliveira H, Xia C. Rigidity of complete minimal submanifolds in a hyperbolic space. *Manuscripta Math* . (2019)158: 21–30. doi: 10. 1007/s00229-018-1013-4

CrossRef Full Text | Google Scholar

9. Fleming W. On the oriented Plateau problem. *Rend Circ Mat Palermo.* (1962)11: 69–90. doi: 10. 1007/BF02849427

CrossRef Full Text | Google Scholar

10. Fischer colbrie D, Schoen R. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Commun Pure Appl Math.* (1980)33: 199–211. doi: 10. 1002/cpa. 3160330206

CrossRef Full Text | Google Scholar

11. Fu H, Xu H. Total curvature and L^2 harmonic 1-forms on complete submanifolds in space forms. *Geom Dedicata* . (2010)144: 129–40. doi: 10. 1007/s10711-009-9392-z

CrossRef Full Text | Google Scholar

12. Hoffman D, Spruck J. Sobolev and isoperimetric inequalities for riemannian submanifolds. *Commun Pure Appl Math.* (1974)27: 715–27. doi: 10. 1002/cpa. 3160270601

CrossRef Full Text | Google Scholar

13. Lawson B. The equivariant Plateau problem and interior regularity. *Trans Am Math Soc.* (1972)173: 231–50. doi: 10. 2307/1996271

CrossRef Full Text | Google Scholar

14. Li A, Li J. An intrinsic rigidity theorem for minimal submanifolds in a sphere. *Arch Math.* (1992)58: 582–94. doi: 10. 1007/BF01193528

CrossRef Full Text

15. Ni L. Gap theorems for minimal submanifolds in $\mathbb{R}^{n+1}$. *Commun Anal Geom* . (2001)9: 641–56. doi: 10. 4310/CAG. 2001. v9. n3. a2

CrossRef Full Text | Google Scholar

16. Neto N, Wang Q. Some Bernstein-type rigidity theorems. *J Math Anal Appl.* (2012)389: 694–700. doi: 10. 1016/j. jmaa. 2011. 12. 013

CrossRef Full Text | Google Scholar

17. Simons J. Minimal varieties in Riemannian manifolds. *Ann Math.* (1968)88: 62–105. doi: 10. 2307/1970556

CrossRef Full Text | Google Scholar

18. Shen Y, Zhu X. On stable complete minimal hypersurfaces in $\mathbb{R}^{n+1}$. *Am J Math.* (1998)120: 103–16. doi: 10. 1353/ajm. 1998. 0005

CrossRef Full Text | Google Scholar

19. Xin Y, Yang L. Curvature estimates for minimal submanifolds of higher codimension. *Chinese Ann Math.* (2009)30: 379–96. doi: 10. 1007/s11401-008-0438-6

CrossRef Full Text | Google Scholar

20. Xia C, Wang Q. Gap theorems for minimal submanifolds of a hyperbolic space. *J Math Anal Appl.* (2016)436: 983–9. doi: 10. 1016/j. jmaa. 2015. 12. 050

CrossRef Full Text | Google Scholar