The black scholes theory and numerical solution finance essay

Finance

Chapter 1

Introduction

Background

An option is an instrument whose value derives from that of another asset; hence it is called a `derivative'. Up until 1973 instruments such as options were priced by guessing. In 1969, Fischer Black an independent finance contractor, Myron Scholes an assistant professor of finance and Robert C. Merton an economist, set out to solve the problem for pricing options. It was in the year 1973 that Black, Scholes and Merton published their analysis in a paper called, “[1] The Pricing of Options and Corporate Liabilities” which was published in the “ Journal of Political Economy.” Scholes and Merton received the Nobel Prize in Economics in 1997; Black was ineligible for the price due to his death in 1995. (Hull J. C (2009) page 277)Prior to the publishing by Black and Scholes, there have been several other publications, dating back to 1877 when Charles Castelli wrote a book called “ The Theory of Options in Stocks and Shares,” which gave an introduction to the hedging of options. The first known analytical valuation for options which used the Brownian motion was written by Louis Bachelier a French mathematician whose dissertation title was “ Théorie de la Spéculation (The Theory of Speculation)” was published in 1900. In 1955 Paul Samuelson an Economist who was a professor at “ Massachusetts Institute of Technology” wrote a paper which was unpublished called “ Brownian Motion in the Stock Market.” In 1962 A. James Boness wrote a dissertation called " A Theory and Measurement of Stock Option Value" which focused more on options. The pricing model which Boness developed was much more of a theoretical jump from the all the publications prior to his, this is the publication which helped Black, Scholes and Merton to introduce their model in 1973[4; page 3]. The Black-Scholes formula changed the way options were priced and made pricing them simple. In 1973 the first option was traded, at the Chicago Board Options Exchange. Options can be traded at official exchanges; in 2004 there were more than 50 official exchanges. (Higham D. J (2004) page 4)

Options

An option is a contract which gives the owner the right, but not the obligation to buy or sell, for a fixed price, the ‘ strike price’. This price is agreed upon the time both parties enter the option. The price of the option is derived from another asset; hence an option is a derivative financial instrument. A person who buys an option is called the ‘ holder’ and the person who sells the option is known as the ‘ writer’. A ‘ call’ option gives the holder the right to buy an asset, for an amount which was specified upon the agreement of the option at a specified time in the future. A’ put’ option gives the holder the right to sell an asset, for an agreed amount which is within a specified time. The amount which has been agreed is known as the ‘ strike price’ or ‘ exercise price’ and time which has been specified is known as the ‘ expiry date’. For a ‘ put’ option if the ‘ underlying price’ is lower than the ‘ strike price’ then the option is known to be in-the-money, hence the value of the option would have the same value if it were to be exercised immediately. The ‘ intrinsic value’ is the difference between the ‘ strike price’ and the ‘ underlying price’, and if this value is negative it’s known as ‘ out of the money’. If the ‘ strike price’ were to be the same as that of the ‘ underlying asset' then this option would be known as ' at the money'. A ' European call option' gives its holder the right (but not the obligation) to buy from the ' writer' a prescribed asset for a prescribed price at a prescribed time in the future. (Higham D. J (2004) page 2) As you are not obliged to buy the shares, you can gain money but you wouldn't lose money. A ' European put option' gives the holder the right (but not the obligation) to sell the ' writer' a prescribed asset for a prescribed price at a prescribed time in the future. (Higham D. J (2004) page 2)Options are very popular, in some cases more money is invested in options than the underlying assets. Investors are extremely attracted to options, to use for in both speculation and hedging. Options can be bought and sold with some confidence as there is a systematic way to find out how much they are worth. Notations which will be used, the price of the stock will be denoted by S, the time by t, the risk-free interest rate by r, the expiry date by T and the exercise price by E. At the time of expiry if S(T)> E then the holder should buy the ' European call option' E and then sell the asset for S(T) hence gaining at profit of S(T)-E, if E≥S(T) the holder of the option gains nothing, then value of a European call option at the expiry date is given by,(1. 1)Figure 1. 1 [2] showing payoff diagram for a European call option. If S(T) ≥E then the holder of the option shouldn't do anything, and payoff of the European put option is,(1. 2)Figure 1. 2 [2] showing payoff diagram for a European put option. If we hold a call option and a put option on an asset which has the same expiry date and the same strike price, the value of when it expires would be,(1. 3)Figure 1. 3 [2] showing payoff diagram for a bull spreadThe diagram above shows a bull spread, which is when you hold a call option with the exercise price of E1, and for an asset and expiry date which is same we write a call option with the exercise price of E2, where E2 is greater than E1. At the time of expiry the value of the first option is max(S(T)-E1, 0) and the value of the second is -max(S(T)-E2, 0, so the overall at the date of expiry is(1. 4)From the diagram above we can see that when the asset price finishes above E1 the holder would benefit but if the asset price were to finish above E2 then the holder wouldn't gain any extra benefit [2; pages 4-5].

Asset Price Model

The movement of an asset price is called a stochastic process. The Stochastic process which is followed by the underlying asset can be either discrete or continuous time, but we will be focusing on continuous time models. When modelling asset prices there are two assumptions, The past history is fully reflected in the present price, which does not hold further information; Markets respond immediately to any new information about an asset. (Wilmott P, Howison S and Dewynne J (1995) page 19)When modelling the asset price, it's mainly about modelling the arrival of information which is new and affects the price. When taking the two assumptions stated above, changes which are unanticipated in the asset price follow a Markov process (see Hull J. C. (2009). page 259.)A Markov process is stochastic process in which everything we know about the future of it depends on the present value. We model the corresponding return on the asset, dS/S by the following stochastic differential equation:

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(1. 5)Where time is given by t, the asset price by S and the change in the given time interval is given by dt, the average rate of growth of the asset price is given by μ and the volatility which measures the standard deviation of the returns is given by σ. We can eliminate the dX term by taking σ= 0, if we were to do this we would be left with the following ordinary differential equation: If μ is a constant then we can solve it to give an exponential growth in the value of the asset, by the following equation:(1. 6)The value of the asset at t= t0 is given by S0. So from this we can deduce that if the volatility is 0, then the asset price will be totally deterministic, and we would be able to predict the price of the asset in the future with certainty. The dX term is the term which contains the randomness, which is a feature of asset prices, and is known as the Wiener process. The following equation is a way to express dX,(1. 7)where Φ is a random variable drawn from a standardised normal distribution. Itô process is a generalised form of the Wiener process (see Wilmott, P, Howison, S. and Dewynne J page 21), and is given by the following formula, which is a more general version of (1. 5):(1. 8)Itô's lemma is used to find the differential of a time-dependent function of a stochastic process. An important relation to the model which connected to (1. 7) is thatWhich just tells us that, the smaller dt becomes the more certain we become dX 2 is equal to dt. When people invest their money they expect a positive return on the investment. When investing, the more risker the investment the more the investors would want in return, as the investor is taking a lot more risk with their money. The returns on this can be shown by the formula:(1. 9)If the asset price follows a random walk had with a

drift[1]of ???????? t and has a standard deviation of ????, then we get a return of that asset of Ri, a proportion of the specific asset can be modelled with the given formula:(1. 10)

The Black-Scholes Model

The Black-Scholes Model was developed under the assumptions of the lognormal dynamics of derivatives. They made the following assumptions [1 page 640]: The asset price follows the lognormally random walk in continuous time. The risk-free interest rate r is known and is constant over time. No dividends are paid for the duration of the option. There are no transaction costs or fees when buying or selling. You can buy or sell and amount of stock even if it’s fractional. You can’t make riskless profit, there’s no arbitrage opportunity. Both the underlying asset trading and change of price is continuous. We use the Black-Scholes formula to find out the price of an option. We need to look at the value at t= 0 when the asset price is S(0)= S0, we look at the function V(S, t) which will give us the option value for an asset price which is S≥0 at the time of t≤T, we assume that the option can be bought and sold at S(0)= S0 at the time t≤T, then in this case V(S0, 0) is the required time-zero option value. Later on we will see the Black-Scholes partial differential equation, for the function of V. The Black-Scholes partial differential equation only valid for the case when V(S, t) gives the corresponding value of a European call or put [2; page 73]. The equation describes the price of the option over time, with this equation you can hedge[2]the option by buying and selling the underlying asset in such a way that we consequently eliminate the risk, hence it implies that there should only be one correct price for the option [5]. The derivation of the Black-

Scholes equation: The price of the stock follows a geometric Brownian motion[3], which is:

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The equation above states that a small rate of return on the stock has the expected value of ???? dt and variance of ???? 2dt. We know what the payoff the option is at maturity, but we need the value of it at an earlier time, by using Itô’s lemma for two variables we deduce:(1. 11)We now construct a portfolio consisting of one option and a number -Δ of the underlying asset. Then the value of this portfolio is:(1. 12)After a small period of dt, and Δ is to remain constant over that period, the change in the portfolio is given by the value:(1. 13)We then put the following equations together, (1. 5), (1. 11) and (1. 12) to get:(1. 14)We can eliminate the random component and make equation completely deterministic by choosing(1. 15)Leaving us with:(1. 16)Now the equation is effectively riskless and the uncertainty has been eliminated. Since we need to earn a return similar to other short term risk-free securities,(1. 17)Replacing this, we get:(1. 18)We then substitute (1. 12) and (1. 15) into (1. 18) and dividing through by dt gives us the Black-Scholes equation:(1. 19)When we solve the partial differential equation in (1. 19) we get an analytical equation for pricing European options, European options can only be exercised at the agreed maturity date, but American style options, which we’ll see later, can be exercised any time up until maturity date. Later we’ll also look at the upper and lower boundary conditions for American and European options.

Chapter 2

Black-Scholes Model

2. 1 American options

American options are similar to European options; the difference between the both is that the holder of an American option can exercise their option at any time which is between the start date and the expiry date. Since the American option gives the holder this option and the European option doesn't, the American option has a higher value. Hence(2. 1)(2. 2)Where the American call option is given by C, put by P and for the European option call is given by c and put by p. Now for the boundary conditions, for the lower boundary conditions which are determined by an arbitrage-free option prices are,(2. 3)(2. 4)The upper boundary conditions are given by,(2. 5)(2. 6)The difference between the price of an American and European options, is usually down which option is in-the-money and the time remaining and also interest rate. An American put which is a dividend paying stock, it is ideal to exercise straight after the dividend has been received, as the payment of the dividend makes the stock price lower and it goes into-the-money. For American and European call options, if the stock doesn't pay dividends the option to exercise early isn't ideal, so the value of the American call option is the same as the European call option. The lower boundary conditions are,(2. 7)(2. 8)The upper boundary conditions are(2. 9)(2. 10)If the stock were to pay dividend, then it could be possible to exercise early and then the value of the American call could be worth more than the European call option. An ' American call option' gives the holder the right but not the obligation, to purchase an agreed asset at an agreed price at any time between the agreed start date and expiry date in the future. An ‘ American put option’ gives the right but not the obligation, to sell the asset at an agreed price at any time between the agreed start date and expiry date in the future. [2, page 173]As American option holders have the option to exercise their option early, so they need to decide when they will be exercising their option if they do. At time t if the option is out-of-the-money[4]then the holder shouldn’t exercise their options, on the other hand if at time t the option is in-the-money[5]they should exercise their option, or they could wait and see if they could get a larger payoff later in time. Now we will consider a call option, letting S(t) be the price of the asset at time t, and letting E be the exercise price, if the holder were to exercise the option at some time where t E at time t. The holder instead of doing this could sell the asset short of the market price at the time of t and then go on and purchase the asset at the time when t= T, and either exercising the option at t= T, or buying at the market price at time T. Using this the holder will gain amount S(t)> E at the time of t whilst paying an amount which is less than or equal to E at time T, which is would be better than gaining S(t)-E at time T. As it isn’t always right to exercise an American call option before the expiry date the value of the American call option will have the same value as a European call option. [2, page 174]

2. 2 Black-Scholes for American options

Now we will look at how we can use the Black-Scholes PDE to find out the value of an American put option. As the Black-Scholes PDE equation follows an arbitrage argument, as is shown next, this argument is only effective for some of the American Options. So the Black-Scholes now isn’t an equation but is an inequality now but only when we exercise early. We will let PAm denote the value of the American put option, with the asset price of S and time given by T. The payoff function is given by(2. 11)Firstly we see that(2. 12)This is true because, if PAm(S, t)< Λ(S, t), we can make a profit quickly by buying the option and straight away exercising it. We know that this particular inequality isn't true for an European put option, as you can exercise the American put early, it makes a difference to the value.[2, page 175]We will look back at the European case, looking at different cases:(2. 13)(2. 14)And we know from (1. 17) thatWe look at case one first, if we buy the portfolio which is V-Π at the time of t, and were to buy the option at V, and then go ahead and sell the portfolio Π. Also by selling the portfolio V-Π at the time of t+Δt, if we were to do this we would certainly make a profit, so case one isn't possible for the European option as we make a riskless profit. For case two, instead of buying as we did for case one, we are now selling the portfolio V-Π at the time of t, so we sell the option V and then go ahead and buy the portfolio Π, and then go ahead and buy the portfolio V- Π at the time of t+Δt, we again would be guaranteed to make a profit so case one and two for the European option isn't possible. [2, page 79]For the American case we similarly consider the following cases:(2. 14)(2. 15)For case one we are selling the asset and then loaning out the case, but for case two we are buying the asset by borrowing the cash. In case of case one, we buy the portfolio which is PAm-Π at time t, so we are buying the option and then we go ahead and sell the portfolio Π, here the we can buy the option and therefore have the control over the exercise facility and arbitrage could be possible hence in this case we are guaranteed to make a profit so case one isn't possible. For the second case, we now sell the portfolio PAm-Π at time t, so we are selling the option and then go ahead and buy the portfolio Π, here we are selling the option so it could be exercised at any time there is no certainty of making profit, so this case is possible. So the Black-Scholes formula we saw (1. 19) now changes to,(2. 16)For the American put, we need to divide the S axis into two different regions for each time of t. When 0≤S≤Sf(t) it is optimal to exercise early so we haveWhen Sf(t) Reference

2. 3 Solution of the Black-Scholes Equation

Now we'll look back the European call option, where the rate of interest and the volatility are constant, when these are constant the solution to a European call is,(2. 20)Here N(•) is given by the cumulative distribution function for a standardised normal random variable, which is,(2. 21)Also d1 is given by(2. 22)And d2 given by(2. 23)The solution for a European put option is given by(2. 24)The European call and put formulas given in (2. 20) and (2. 24) are quite popular in finance as these formulas can be used easily, because excluding volatility all the other parameters can be observed from the market. So now we will look at methods for working out an estimate for the volatility.

2. 4 Volatility

Volatility is a standard deviation measure which is the assets potential of deviating from its current price. The value volatility is greater for a specific option, for which the greater the volatility of the underlying asset the greater the value of the option will be. In the case of options volatility is good compared to that of other financial assets, this is because the person buying the option gets only the upside potential and doesn't get the downside risk, whilst for other assets they have both risks. When investors are investing they are usually expected to be risk averse and they would place a lower value on an asset which has a highly priced volatility asset, as volatility gives us an uncertain value which means they could be at risk of making a loss. The value of volatility, which is in asset markets, is mainly caused by the information released. The information released falls into two different categories; the first one is anticipated information and unanticipated information. Social and political information also economic statistics are examples of anticipated information; this information is usually driven by the expectation of the markets, also it can be analysed with reference to previous resources, because we can make probabilistic expectations of an anticipated asset price volatility using past response of the market. The other one is, unanticipated information examples of this is natural disasters and wars, these could have a considerable effect on asset price volatility, and it's really hard to predict when this will happen. Implied volatility is when we you extract the volatility from the observed market data, which has a given option value, as well as knowing the values of the underlying, the exercise price and the time to expiry, once we know these values we can work backwards to work out the volatility, then once we have the volatility we can then use the Black-Scholes formula to work out values of other options on the same asset. There's more advanced ways in which we could calculate the market of volatility where we use more than one option price, we can work out market's view of future values for the volatility of the underlying. Implied volatility has an unusual feature in that it doesn't usually appear to be consistent across exercise prices. An option which has different strike prices but with the same maturity, would have different volatilities and this is called a volatility smile. The volatility smile has the general form shown in figure (1. 4), the value of implied volatility is quite low for at-the-money options, and then it becomes increasingly higher as the option moves closer to into-the-money or out-of-the-money. We can use the Newton-Raphson[6]method numerical method can be used with the derivation of the volatility with respect to the option price. [2][3][8]http://www. theoptionsguide. com/images/volatility-smile. gifFigure (1. 4) [8] showing the volatility smile. We have now seen how you can work out the volatility using implied volatility, but now we will look at another method to work out the volatility, using the historical method, in this method we estimate the volatility by using the pervious behaviour of the assets. In this method we have to calculate the standard deviation of the logs of the price changes of a certain sample time series of a historical data for the asset price. When we estimate the volatility using this method, we have to observe the stock price at fixed intervals of time; they could be every day, week or month.(2. 25)Where n+1 is the number of observations, Si is the stock price at the end of the ith interval and t is the length of time which is in intervals in years. For the variance, it is estimated by a sample variance and is normalised by (n-1), so it's an unbiased statistic(2. 26)If the asset were to pay dividends, then we would have to change the asset price sequence to reflect the non-homogenous nature for the data series, as a dividend payment increases the the return to be paid to a buyer, if the dividend D was paid on an asset then Ui would change to(2. 27)When calculating, the return on the other time intervals that don't pay dividends, the return is still (2. 25). When estimating the volatility, the days in which trading doesn't occur is also included, to overcome this we ignore the days in which trading is closed and so when calculating volatility per annum we use(2. 28)So when we are calculating the life of an option, we use trading days rather than all the days in the year. The life of the option also changes and the new value of T is given in (2. 29) [3]

(2. 27)

2. 5 Dividends

So far for the Black-Scholes models we have assumed that no dividends is being paid, but now we will see how we can change the Black-Scholes model to include the payment of dividends. Firstly we have to assume that the amount and the timing of the can be predicted during the life of the option, but the only problem with this is that you can only do this with long-term options as it's quite difficult to assume with the short-term options. Firstly we will consider a payment structure, where dt is the time, D0Sdt is the amount of dividends that the underlying asset pays out, where D0 is a constant. This payment structure is independent on time apart from the dependence is on S. We will consider a dividend yield which is the amount of asset price which is paid put per unit time. So the dividend D0Sdt is the constant and D0 is the continuous dividend yield. Now that we have this information we will look at how dividend payments affect the asset price, as the value of the share is lowered after the payment of the dividend the expected return will change to (????-D0) and so the asset price which is given in (1. 5) changes to(2. 28)Now we will look how this will affect the Black-Scholes PDE(1. 19), we assume that the dividend will not have effect on the option price as the PDE isn't effected dt in the differential for S, but this isn't the case as we receive a payment of D0Sdt for every asset held, as we hold -Δ of the underlying asset, then are portfolio will change by the amount of - D0SΔdt and so the change in portfolio(1. 13) will now change to(2. 30)Due to this change the new Black-Scholes PDE we see that(2. 31)Looking now at the solutions of the model, we see the that for a call option the final condition still stays the same and is C(S, T)= max(S-E, 0), also the boundary condition is the same at S= 0 still remains at C(0, t)= 0, but there is one change to the boundary conditions that is when we use the Black-Scholes model in (2. 31) and it is(2. 32)2. 32 is true, due to the fact that as S→∞ the value of the option will be becomes equal to the asset, but this is without dividend income. When calculating the value of the option with dividends we do it as we did before with dividends, and let(2. 33)From the above equation (2. 33) we see that it can satisfy the Black-Scholes equation (1. 19) but the only difference is that r is being replaced by r-D0, and so the value of C1(S, t) is like the normal European call option, but with a different interest rate of r-D0, and so the value the European call option now is(2. 34)And the European put option is given by(2. 35)Where (d10) and (d20) are given by(2. 36)(2. 37)The values given above are similar to that of the ones we achieved earlier in (2. 22) and (2. 23) but the interest rate has now changed from r to (r-D0). [8]Now we will look at American call options, we know from before that if there weren't any dividends then the option shouldn't be exercised early, but now when there are dividends it can only be ideal to exercise at a time straight before the stock goes ex-dividend. From before we know that C(S, t) of the call satisfies (2. 31) if the exercise isn't ideal. The payoff condition is given by(2. 38)but due to the early exercise option, we always get(2. 39)If there were to be an optimal exercise boundary at S= Sf(t) then(2. 40)If the exercise boundary were to exist then the Black-Scholes formula in (2. 31), will only be valid for when C(S, t)> max(S-E, 0), but max(S-E, 0) isn't a solution of the Black-Scholes equation in (2. 31), but to accommodate for this we can modify the Black-Scholes equation in (2. 31) but an inequality, so we get(2. 41)for which the equality hold only for when C(S, t)> max(S-E, 0). If it were optimal to early exercise, it would be because the value of the option would be less valuable than if it were to be held than it were to be exercised straight away and funds were to be deposited in a bank account where interest is paid.[8]

2. 6 Options on Futures

Firstly we will look at forward and future contracts, these are meant to be easier to value than options, due to the fact that the risk involved can be removed by a one of hedge at the start of the contract. As a result of this, it is possible to value them independently of any assumptions which are made about the behaviour of the price of the asset, but this only if we can predict the interest rates. We need to note that future and forward prices are the same. The forward price is calculated at the start, separately for each of the contracts. We will let F be the forward contract, now we will try and find a relationship between S(t) and F, so that there is fair value for both parties, and we assume that the interest rate stays the same throughout the contract. There are different ways in which we can find out the forward price; firstly we will look at one which is based on arbitrage. If there was a party who is on long contract, which had to deliver the asset at the time of T, but they don't know what the asset price will be at T, but this is not a concern as they can satisfy their part of the contract, by first borrowing the amount S(t) at the start of the contract, once you've brought the asset you can make money, when you exercise and this money is used to pay off the loan, the interest rate is given by r, and so the amount the loan will cost is S(t)er(T-t) and hence the forward price is given by(2. 42)Now we will look at a party which is a long contract, a simple way of looking at this one is that a long position in the forward contract is the same as a long position in a European call option, and the short position the same as European put option, both of which have the same expiry and exercise date to that of the forward contract. As the forward price has no value at the start, then the exercise price E, which is the same as the forward price F will give us(2. 43)Finally, the forward contract is a derivative contract, hence it must satisfy the Black-Scholes equation, the payoff time which is given at time T, is S-F and so we can work out the solution at an earlier time of t using(2. 44)Up until now, we have assumed that no dividends were paid, if the contract was to pay a constant dividends of D0 then the equation in (2. 42) changes to(2. 45)Options on futures or future options are contracts where the exercise of the option would give the holder a position in a future contract. A futures option is a contract which gives the holder the right but not the obligation to buy or sell a futures contract at a fixed price which is called the strike price, which is up to an expiration date which is specified. A put future option gives the holder the right to enter into a short futures contract at a specific price, whilst a call futures option gives the holder the right to enter a long futures contract at a specific price. A futures option can be an American style option or a European style option. If a call future option were to be exercised the holder would get a long position in the underlying futures contract and also a cash amount which is the same as the most recent settlement futures price minus the strike price. If a put futures option were to be exercised the holder would get a short position in the underlying futures contract and also a cash amount which is equal to that of the strike price minus the most recent settlement futures price. [3][8]

2. 6. 1 American Options on Futures

The intrinsic value is the minimum value of an American call and American out on futures. The intrinsic values for a call given by C and a put given by P, where F is the futures price and the strike price is given by E, are(2. 46)(2. 47)We know from before that if there isn't no dividends paid on the stock for a put option it could be optimal to exercise early but for a call option it wouldn't be optimal, but for a futures contact both a call and put option could be exercised early. If the strike price and future price were to be the same then the two options would be equivalent.

2. 6. 2 European Options on Futures

For the European options on futures, as they don't have an option to be exercised early like American options, this suggests that the value of time is zero this is due to the fact that the European option can only be exercised at maturity which is at time T. The boundary conditions are given below(2. 48)(2. 49)

2. 6. 3 Black's Model

During 1976, Fischer Black created a different version of the Black-Scholes model for the pricing of European options on futures, but we have to assume thatThe futures and the option are to expire at the same timeThe future price must be equal to the forward price, for them to be equal we also assume that the interest rates are non-stochastic. The modified Black-Scholes formula for a European call option is given by(2. 50)And the put option is given by(2. 51)The values of d1 and d2 are given by(2. 52)(2. 53)The Black formula above is similar to the Black-Scholes formula for pricing options, but spot price of the underlying asset is swapped with the futures price [6].

Chapter 3

Finite Difference Methods

The finite difference methods try to solve the Black-Scholes Partial differential equation (PDE) and linear complementarity problems. To solve them, we need to approximate the differential equation over the area which being integrated by a system of algebraic equations. We be looking at means in which we can obtain the numerical solutions to the PDE's, these also contain very powerful techniques which are flexible and are capable of generating precise numerical solutions to the PDE's. We will be looking at three different ways to solve the Black-Scholes Partial differential equation; these are the Explicit method, the fully Implicit method and the Crank-Nicolson method. The three different methods are similar but they have different accuracy, the amount of time it takes to execute each ones is different and also the stability differs from each one. When we solve the PDE's, we need to look at three different components, the PDE, the region in which the PDE has to satisfy and the boundary and initial conditions which need to be met.

3. 1 Discretization of the Equation

When doing the finite difference method, we will be discretization the partial differential pricing equation and also the boundary conditions by using a forward or a backward difference approximation. We can write the Black-Scholes Partial differential equation in (1. 19) as(3. 1)We will be discretizing the equation with respect to the underlying asset price and time. First we will divide the (S, t) plane into a suitably dense grid, and then approximate the minute steps ΔS and Δt by using small fixed finite steps. The life of the option will be given by T, we then divide this into N equally spaced intervals of length Δt= T/N, so in total N + 1 times are considered, 0, Δt, 2Δt,..., T. We let Smax denote a stock price which is really high so that when it's reached the value of put has virtually nothing. We define ΔS= Smax/M we will look at total of (M+1)(N+1) points which is shown in the Figure 3. 0. From the grid, the (i, f) point is the point on the grid which corresponds to the time of iΔt and the stock price of jΔS, we will let fi. j be the variable which will denote the value of an option at the point (i, j)[3]. Figure 3. 0 [3] showing mesh points for the finite difference approximation.

3. 1. 1 Finite Difference Approximations

For this part we will be replacing the partiadl derivatives which occur in the PDE's by using approximations based on the Taylor series expansion. We will let f(t, S) represent the grid above, and so the expansions in Taylor series for f(t, S + ΔS) and f(t, S - ΔS) are(3. 2)(3. 3)Now we will use (3. 2) to show what the difference equation is(3. 4)Now we will use (3. 3) to show what the backward equation is(3. 5)We now go ahead and minus (3. 3) from (3. 4) and take the first order partial derivative to give us the central difference equation(3. 6)Next we will sum (3. 2) and (3. 3) and take the second order partial derivative(3. 7)Now we will expand using the Taylor series f(t+Δt, S)(3. 8)And finally the forward difference equation for time is(3. 9)If we were to replace both the first and second derivatives in the Black-Scholes Partial differential equation, we would get a difference equation which will give us an equation that we can use to get the solution f(S, t) [3].

3. 1. 2 Boundary and Initial Conditions

If we were to have a PDE without any initial conditions or any boundary conditions, then we could possible get no solutions or infinity amount of solutions. The put option value is given by max(E-ST), 0 where the stock price is given by ST the time by T and so(3. 10)And so when the stock price is zero, the value of the put option is K,(3. 11)When S= Smax we assume that the value of the put option is zero(3. 12)The equations in (3. 10), (3. 11) and (3. 12) give the value of a put option, where S= 0, S= Smax and t= T [3].

3. 1. 3 Log Transform of the Black-Scholes Equation

Now we look at the log transform, if we let S be the stock price, then we could use instead of S, lnS for the underlying variable when we use it to apply to the finite difference methods. This is due to, when σ is constant then the standard deviation of lnS is will also be constant. We will let y= lnS and f(t, S)= g(t, y), which is the price of the call at time t, we now work out the call price in terms of the time and the log of the asset price.(3. 10)After differentiation we get(3. 11)And for time we get(3. 12)Now the Black-Scholes Partial differential equation in (3. 1) can be written differently, as we substitute the y and t with the terms (3. 10), (3. 11) and (3. 12) to obtain(3.

13)We divide the grid into finite intervals with yM equally spaced M+1 grid points and with N+1 equally spaced points tN for time. We assume that the stock price is log-normally distributed and so can have a maximum of infinity and minimum of zero, this is because lnS gets closer to -∞ as S gets closer to zero, and so need to let ω be small so that lnS=�" for S <1, so that we can avoid negative stock prices.

3. 2 The Explicit Finite Difference Method

This method is advantageous because it's very robust and it converges to a solution to the differential equation as Δt and ΔS approach zero. One of the disadvantages is that M-1 simultaneous equations would have to be sold for us to calculate the fi, j from the fi+1, j. If we were to the value of the option at the maturity time, then it could be possible to give an expression which would give us the next value fi, j clearly in terms of the given values fj-1, j+1, fj+1, i+1. We can simplify this method by assuming ∂f/∂S and ∂2f/∂S2 at the point (i, j) on the grid is the same as at the point (i+1, j), as we have assumed this the equation given in (3. 6) and (3. 7) become(3. 14)(3. 15)Then the difference equation is given by(3. 16)We can re-arrange this to give(3. 17)The weights of ???? 1, ???? 2 and ???? 3 are

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The weights given are the risk neutral probabilities of S-ΔS, S and S+ΔS. all at t+Δt. For this method to work well all three probabilities should be positive. So if they are negative probabilities we have a problem as this produces results which will not converge to a solution for the differential equation [3].

Project Plan

Weeks 3-8During week 3 I will be given my dissertation topic and know what dissertation I will be doing. I will try and see my supervisor on a weekly basis so that I have regular feedback on my dissertation. Early during this period I will do some background research on, Options, Black-Scholes and Asset price model. To do this research I will need to look for books, which will help me for my background information and during the dissertation. During week 8, the background, project plan and Gantt chart will be due in. Weeks 9-12During this 4 week period I will do more research on Options, specifically American style options and look at upper and lower boundary conditions for both American and European style options, and do the write up for these topics. Weeks 13-14During these two weeks, which are holidays, I will start looking into detail the Binomial Model, and see what effect it has on valuing options. Also I will start doing research into Finite Difference Methods. Weeks 15-19I will start doing in depth research in to Finite Difference Methods. Also during these 5 weeks I will start implementing on Matlab. Weeks 20-24In these weeks, I will analyse the results I get from Matlab and complete my final write up. The dissertation is due in on week 25, but I've left an extra week for anything I've left out or any final checks. Weeks 25-29Week 29 is when the oral presentation is, so in these weeks I will prepare my presentation and rehearse for it, also go over my dissertation as I will be getting asked questions.