Problem solving in school mathematics

Problem solving has an important place in the world of mathematics. Branca (1980, p. 3) quoted Lester 1977 , 'problem solving has been said to be at the heart of all mathematics' to illustrate the importance of problem solving. However, in the field of school mathematics the primary goal of teaching mathematics is to develop the ability to solve a variety of mathematical problems. Monaghan, Pool, Roper, & Threlfall (2009, p. 21) states that according to Lester (1994, p. 661) 'most mathematics educators agree that the development of students' problem-solving abilities is a primary objective of instruction' but according to Schoenfeld, problem solving has 'multiple and contradictory meanings' . Monaghan, et al.(2009) raise the question 'What is the use of students learning mathematics if they cannot use it to solve problems?' (p. 21) and continue to say that this gives a signal that more problem solving needs to be done in schools and that students struggle in solving simple problems even when they are directly related to the mathematics that they have learned. Thus, research on problem solving in mathematics has been done to find strategies on how to solve problems.

My main focus in this essay is to relate Polya's, Burton's and Schoenfeld's approaches on how to solve problems with the aim of finding the most suitable approach. First I will define what a mathematical problem is. Then I will discuss types of mathematical problems and numerous researches done on mathematical problem solving over the past decades. Next I will briefly illustrate steps on each of the processes: Polya's, Burton's and Schoenfeld's episodes. Finally I will relate the three approaches with the aim of finding the most suitable approach.

2. What is a problem in mathematics?

In this section the definitions of 'mathematics problems' will be discussed with the aim of providing information as to what a mathematical problem is.

Frobisher (1994, p. 154) describes a problem as a 'situation that has interest and appeal to a child, who therefore wishes to explore the situation more fully in order to gain understanding of it. Goals arise naturally during the exploration and are determined not by the setter of the problem but by the child. The child in turn surveys the problem situation before exploring avenues of interest, following paths which may or may not lead to a satisfactory conclusion. As Ernest(1991) so succinctly puts it, 'the emphasis is on the exploration of the unknown land rather than a journey to a specified goal' '.

Kilpatrick(1985, p. 2) defines a problem as ' a situation in which a goal is to be obtained and a direct route to the goal is blocked'.

R. Mayer (1985, p. 123) states that 'A problem occurs when you are confronted with a given situation- let's call that the given state - and you want another situation - let's call that the goal state - but there is no obvious way of accomplishing your goal'. Furthermore, Mayer(1985) gives the example of finding the volume of a frustrum of a right pyramid where the value of the sides of the two bases and the height is given. ' If you did not know a formula for volumes of frustrums, this would be a problem for you (Polya, 1965)' (R. Mayer, 1985, p. 123).

Orton & Frobisher (2005, p. 25) says ' a mathematics problem for one learner may be an exercise for another' because if a student has had a similar situation before the student may consider the problem to be an exercise, where another who has never come across a similar situation may recognize this as a problem. He describes 'a mathematical problem can be said to be a situation in which an individual student:

Recognizes or believes that there exists a mathematical goal to be achieved, usually an answer of some kind;

Accepts the challenges to perform some mathematical task in order to reach the goal;

Has no readily known or recallable mathematical procedure available to enable the goal to be attained directly' (p25)

In Frobisher's (1994) definition he mentions that the goal is not determined by the setter (teacher) but by the student. This statement is not valid for all problem types. In some problems such as classroom word problems the student does not determine the 'goal' (answer). In some questions different paths can be taken to obtain the desired 'goal' (answer) which is determined by the teacher. Furthermore, in problem types like investigation problems the children could obtain different answers (goals).

It can be concluded by the above given definitions that a mathematical problem can be a situation in which a student recognizes the existence of a mathematical 'goal', performs some mathematics to achieve the goal although there is no known or recalled mathematical procedure for getting to the final 'goal' which is usually the answer.

3. Types of mathematical problems

According to LeBlanc, Proudfit, & Putt (1980, p. 104) standard textbook problems (word problems) and process problems are the two main problem types that are widely used in school mathematics. The characteristics of the two methods will be explained broadly.

3. 1 Word (standard text book) problems

Word problems are commonly used in elementary school mathematics textbooks and mathematics books. Frobisher (1994, p. 152) states that in a word problem a task is presented in words or symbols, and a goal is set by asking a question. According to LeBlanc, et al. (1980, p. 105) the main characteristic of a word problem is that it can be solved (achieving of the goal) by using previously mastered algorithms or operations. He further states that the purposes of the word problems are to improve the ability of recalling factors, strengthening of the skills with operations and algorithms and building the relationship between operations and their relationship between real world situations. The main factors that affect the difficulty of these types of problems are the level of mathematics, the complexity of the algorithm and the number of steps involved in solving. (LeBlanc, et al., 1980)

Orton & Frobisher (2005, p. 27) describes 'Routine problems' which is a category of word problems that 'uses knowledge and techniques already acquired by a student in a narrow and synthetic context'. The example given below is a routine problem.

'How many more than 432 is 635?' (p. 27)

In these type of questions a student is expected to understand the 'linguistic complexity' and change it to a model with mathematics symbols or operations. Story problems are also a type of word problems that are set in a ' real context' which often needs an understanding of the real world situation. An example of a story problem is given below:

'A postman has ninety four letters in his bag.

Twenty five of them are first class. How many are second class?' (Orton & Frobisher, 2005, p. 27)

In the above problem the understanding of the different classes of letters is essential for a student to solve it.

As word problems use the facts and techniques taught recently, it is ' not an appropriate method for development of new knowledge and its contribution to mathematics knowledge is minimal'(Orton & Frobisher, 2005). Furthermore, Orton & Frobisher (2005) argues that word problems are mathematical exercises rather than questions since they do not satisfy the criterion that there is 'no readily known or recallable mathematical procedure available to enable the goal to be attained directly'. Nevertheless, I feel that the answer ('goal') is also predetermined by the question setter .

3. 2 Process problems

Process problems are also type of problems that appear in mathematics textbooks , but not available to elementary school students in Sri Lanka. Unlike in word problems this type requires 'strategies and non algorithmic approaches'(LeBlanc, et al., 1980, p. 105) and often has more than one answer.

According to LeBlanc (1980) more emphasis is given to the process of obtaining the solution rather than the final solution. Moreover the success of solving the problem depends on the use of one or more strategies and not on the ' application of specific mathematical concepts, formulas or algorithms'(LeBlanc, et al., 1980, p. 105). Orton & Frobisher (2005) takes a similar view and says that children who reflects on different processes develop the ability to solve other problems.

LeBlanc, et al.(1980, p. 105) states that process problems 'encourage the development and the practice of problem solving strategies.., provides an opportunity for students to devise creative methods of solution, to share their method with other students , to build confidence in solving problems … and to enjoy mathematical problem solving' . Furthermore, he explains that the difficulty of process problems depends on the number of conditions that must be satisfied, the complexity of the conditions and the type of strategy used by the solver .

Butts (1980) in Orton & Frobisher (2005, p. 158) describes 'open search type of problems' as 'one that does not contain a strategy for solving the problem in its statement. Thus, the openness refers to the method of solution, not to the solution'. He gives the following example:

How many different triangles with integer sides can be drawn having a longest side(or sides) of length 5cm? 6 cm? n cm? In each case, how many of the triangles are isosceles?

As it does not have one specific path to the solution I feel 'open search problems' are a type of process problems.

At this point, I would like to express my ideas about these two problem types. Firstly, in the elementary school level where students engage more on the basic operations and algorithms it is appropriate to use word problems as they help the children to familiarize themselves with the mathematical concepts. With my former experience as a mathematics teacher I have observed that when students do not master in their basic mathematical skills in lower level classes they are unable to solve complex problems or think innovatively when they come to higher classes. More seriously students come to a conclusion that 'Only geniuses are capable of discovering or creating mathematics'(Schoenfeld, 1997). Whereas, the actual difficulty is the lack of basic mathematical knowledge. On the other hand, I believe that as the students come to higher grades it is essential to engage in process problems as it broadens the students mathematical knowledge and the reasoning abilities.

4. Research on mathematical problem solving

According to Suydam (1980, p. 35) early research on mathematics problem solving has focused mainly on word (textbook) problems. The main emphasis was on how children solved problems. In doing so it was that a way could be found to teach problem solving. It was during this period that Polya produces How to Solve It (1954) 'a charming exposition of the problem-solving introspection' (Schoenfeld, 1987, p. 17). Every researcher since then has based their research on mathematical problem solving on Polya's work.

According to English, Lesh, & Fennewald (2008) most researchers have tried to investigate the questions: '(a) Can Polya-style heuristics be taught? (b) Do learned heuristics/strategies have positive impacts on students' competencies?' but it was not successful.

Begel's (1979) and Silver (1985) in (English, et al., 2008) have concluded after the review of literature in mathematics education that there is little evidence that transfer of learning has been successful. Though it has been reported in some studies that successful learning has occurred, Silver suggests that it is due to students mastering the 'mathematics concept, rather than from problem solving strategies, heuristics or problem solving process'.

Schoenfeld (1992, p. 53) states 'Pólya's characterizations did not provide the amount of detail that would enable people who were not already familiar with the strategies to be able to implement them'. He also says that the reason for the lack of success was due to Polya's heuristics being 'descriptive rather than prescriptive'. According to Schoenfeld in English, et al. (2008) problem solving research should help students to develop a larger number of 'specific problem strategies', learn meta-cognitive strategies and to 'improve beliefs about nature of mathematics , problem solving and their personal competencies'.

Lester Koehle(2003) in (English, et al., 2008) states that even ten years after Schoenfeld's proposal, research on problem solving has failed. According to him 'Schoenfeld's proposal simply moved the basic heuristics to a higher level' not achieving 'prescriptive power'. However, as a response to this

Schoenfeld at the 2007 NCTN Research Pre-session proposed that researchers should focus on 'meta-meta-cognitive processes'(English, et al., 2008). But as in the earlier cases this had the same shortcomings and was remarked as a 'short list of descriptive rules that lack prescriptive power and a longer list of prescriptive rules involve knowing when and why to use them' (English, et al., 2008).

Finally, as the field of research has failed more than 50 years, Lesh and Zawojewski(2007) concludes that it is time to 're-examine the fundamental level of assumption' and suggests that an alternative is to use theoretical perspectives and use methodologies such as 'models & modeling perspective (MMP) on mathematics problem solving (English, et al., 2008).

5. Polya's Four Phases

Polya (1973) grouped his observations into four phases (understanding the problem, planning, carrying out the plan and looking back) describing the four stages the person passes through during the problem-solving process. These four phases would be a guideline to solve a problem. Let us examine them separately by using a fifth grade mathematical problem: question 1 (appendix 1).

5. 1 Understanding the problem

The student should be able to point out the unknown, the given data and the conditions. The teacher's role at this point is to ask questions and guide the student. 'Some probable questions the teacher can ask are: What is the unknown? What are the data? What is the condition?' (Polya, 1973).

Depending on the question sometimes it is easier to understand if the student draws a figure and use notations.

As an example: in question1 (appendix1) lines 1- 6 shows that the student tries to understand the question by gathering information about the problem. 'Can you tell in your own words what the problem is asking you to find?' (line 6) show that the teacher checks whether the student understood the question by 'asking the student to repeat the statement' (Polya, 1973, p. 6) in his own words.

5. 2 Devising a plan

Mayer (1983, p. 44) states that in this phase the solver tries to use prior knowledge and experiences to find a method of solving the problem. Planning is needed to decide which calculation or construction should be done to obtain the unknown answer. According to Polya (1973) it is often easy to look at a formally solved problem and try to relate to it or think of the acquired mathematical knowledge or even present a problem which has been solved before and ask the students to use it. The student can either find out whether he can 'restate the goal in a new way based on' his prior experience ('working backwards') or whether he can 'restate the givens in a way that relates' to his prior experience ('working forward')(R. E. Mayer, 1983, p. 44).

As an example: in question1(appendix1) lines 7-9 the students figure out different methods they think would be more suitable to approach the answer.

5. 3 Carrying out the plan

According to Polya (1973) when it comes to this step because the most critical step in planning is over it is just a matter of carrying out the plan patiently. If the student is convinced of the plan he may carry out the plan smoothly, but in an instance where he has reached the plan by an outside source he might forget the plan. Therefore, it is important that the student understands the steps of the plan. The teacher too has to insist that the student 'check each step' (Polya, 1973, p. 13).

As an example: in question1(appendix1) line 10 show the students carrying out the formally planned steps.

5. 4 Looking back

Polya(1973) states that it is important to re-examine and reconsider the way they solved the question and the answer. It allows them to rethink about the question and the method of solving and sometimes to come up with a different, easier method. By doing so, the children learn to believe in themselves. The teacher could extend the problem so that the student could see how the method fits to another problem or for the student to obtain deeper understanding. Shumway (1982, p. 134) states 'one could argue problem solving ends and the concept learning begins when one begins looking back, identifying similar problems, and engaging in other post-solution activities'.

In question1(appendix1) lines 11-26 clearly shows the discussion between the teacher and the students which enables the students to reflect back on their answers. 'When we draw up a list like Stephens' and put the numbers in some order, we call it an organized list' (line 24) shows that through discussion they decide which is the easiest and the most appropriate method. Moreover, in lines 25-26 the teacher continues the discussion so that the student forms a broader understanding.

6. Burtons' three phases

According to Burton three phases of activity namely Entry, Attack and Review(L Burton, 1984) can be observed in the process of problem solving. Let us examine each phase separately using the same question used above.

6. 1 Entry

This is the first step where the solver tries to understand the questions as to what it is about and what needs to be done. Mason, Burton, & Stacey (1982, p. 29) suggests to answer the three questions: 'What do I KNOW?'

'What do I WANT?'

'What can I INTRODUCE?'

According to Mason, et al (1982) in this stage the question needs to be read carefully and the relevant facts need to be extracted from the question. The solver also can restate the question in his own words so that he understands what needs to be discovered. Introducing diagrams, symbols, images and notations not only makes it easier to obtain a clear picture but also makes it easier for the next phase.

As an example in question 1(appendix1) lines 1-7 falls under this phase and can be rephrased using the key words.

'I WANT' to find the different coins Susan could use to pay for her candy. I KNOW' the cost of a candy bar. 'I KNOW' which coins the machine takes. 'I KNOW' Susan cannot pay with quarters because the machine does not accept quarters. 'I KNOW' there are more than one answer to this question so 'I WANT' to find different ways of paying. 'Can I INTRODUCE' a table , draw the different coins or write down the denominations?

6. 2 Attack Phase

It is at this stage that the question starts to resolve. In the attempt to solve the question the solver might take several approaches or different plans may be executed. Mason, et al.(1982) states that it is quite normal for the solver to get 'stuck' and therefore re-think of a different plan or even go back to the entry phase . 'Stuck' is always accompanied by 'aha!' which means a different approach or a way out.

When the solver is 'stuck' 'TRY'….., 'MAY BE'……, 'BUT WHY'……. (Mason, et al., 1982, p. 77) may help him to overcome the problem. However, this phase is complete when the problem is resolved.

In question1(appendix1) in lines 8-10 the children get entries for the table by trying different combinations: Let us 'TRY' 4 nickels and 5 pennies. 'MAYBE' we should 'TRY' all possible ways with 2 Dimes…… 'BUT WHY' can't we use 3 quarters?

6. 3 Review Phase

As the name suggests, it is the step where you look back at what you have done, try to extend it to a higher level or improve it. The three words that help to structure this phase:

'CHECK'… the calculation, arguments to verify that the computations are appropriate, that the resolution is correct

'REFLECT'…on the key points, arguments, resolution

'EXTEND'…the results, by seeking a new path , by altering some of the constraints. (Mason, et al., 1982, p. 47)

In question 1(appendix 1) in lines 11-26 the students will 'CHECK' the answers and the teacher compares Becky's and Steplen's answers.

'REFLECT' on the key point like where Stephen says, " but with dimes first, then nickels and then pennies." 'REFLECT' on putting the numbers in some order. They can 'REFLECT' on different results of other groups.

To 'EXTEND' the problem the teacher or the students could ask: if a candy bar cost 30 cents show the ways Susan could put coins into the same machine. 'How does the number of ways alter if the machine will also take quarters?'

7. Schoenfeld's Episodes

Schoenfeld (1997) has parsed protocols into episodes for the purpose of understanding the process. These are periods of time where the solver engages in similar actions. Episodes include reading, analyzing, exploration, planning/ implementation and verification. We shall try to explore the episodes using the previous question.

7. 1 Reading

This episode starts when the solver starts to read the problem. This includes the time he spends to understand the problem and the rereading of the problem.(Schoenfeld, 1997)

In question1 (appendix 1) the children would read the problem, may be read it again and try to understand the question.

7. 2 Analyzing

According to Schoenfeld (1997) in this stage the solver tries to fully understand the problem by selecting an appropriate perception, re organizing the question accordingly and introducing the mechanisms that might be suitable for the problem. In cases where the solver knows the relevant perspective this episode may be bypassed.

In question1 (appendix 1) lines 1- 6 can be listed under this section . At this point the student tries to understand the question with the objective of finding an answer.

7. 3 Exploration

This is where the solver explores for relevant information that can be used in other phases. This is less structured than analysis and loosely related to the conditions and the goals of the problem. (Schoenfeld, 1997)

7. 4 Planning/ implementation

In these stages issues such as whether the plan is well structured, whether it can be orderly implemented, whether its progress is being observed and reported to the solver is addressed. (Schoenfeld, 1997)

In question1(appendix 1) the lines 7-10 falls to this category where the students plan and implement it.

7. 5 Verification

This is where the solver revives the solution. Some questions that are used in this episode are: 'Does the problem solver review the solution?, Is the solution tested in anyway? If so, how?, Is there any assessment for the solution?'(Schoenfeld, 1997, p. 300)

Lines 11-24 in question1 (appendix 1) shows that the teacher and the students verify the answers they obtained.

8. Protocol analysis

This section contains the parsing of the protocol 1 (appendix 2) using Polya's and Burton's and Schoenfeld's approaches. The analysis using Polya's and Burton's Phases were done by closely observing the child's work which is exhibited in the protocol 1 and the parsing by Schoenfeld's episodes were extracted from Schoenfeld (1997).

8. 1 Protocol Analysis using Polya's Phases

understanding

Item 1- 4

understanding and planning

items 5 - 8

Planning and carrying out the

plan

Item 9-19

Look back

Item 20-21

Planning and carryout the plan

Item 54-55

Look back

Item 56

T1: (Item 2)

T2

T3

T4: (Item 22)

8. 2 Protocol Analysis using Burton's Phases

Entry Phase

Item 1- 8

Attack Phase

Item 9 - 19

Review

Item 20 -24

Entry & attack

Item 25- 39

Review and attack

Item 40 -48

Attack

Item 54-55

T1: (Item 2)

T2

T3

T4: (Item 22)

Review

Item 56

8. 3 Protocol Analysis using Schoenfeld's Episodes(Schoenfeld, 1997, p. 311)

E1: Reading

Item 1

(1 minute)

E2: Analysis

Items 3 - 8

(2 minutes)

Local Assessment: Item 3

Local Assessment: Items 7, 8

E3: Planning - Implementation

Items 9 - 19

(4 minutes)

Local Assessment: Items 15, 16

Local Assessment: Item 18

E4: Verification

Items 20, 21

(30 seconds)

E5: Analysis

Items 22 - 39

(4 minutes)

Metacomments: Items 24, 25

(Meta) Assessment: Item 33

Local Assessment: Item 39

E6: Analysis

Items 40 - 48

(3 minutes)

Local Assessment: Item 43

Local Assessment: Item 48

E7: Exploration

Items 49 - 53

(3 minutes)

Metacomments: Items 49, 50

E8: Analysis - Implementation

Item 55

(35 seconds)

E8: Verification

Item 56

(1 minute)

T1: (Item 2)

T2

T3

T4: (Item 22)

T5: (Item 39)

T6: (Item 49)

T7: (Item 54)

T8

9. Relating the three Approaches

Under this section the protocol analysis of Scheonfeld's episodes will be compared with Polya's and Burton's Phases. As all three approaches have analyzed the same protocol by doing so I wish to compare the three processes against each other and to identify how well each approach fits a real problem solving situation.

9. 1 Relating Polya's phases and Scheonfeld's Episodes.

According to the protocol analysis using the two models Polya's Phases(1973) and the Scheonfeld's Episodes(1997) the following observations can be made.

Line 1- 4, shows that the student reads the problem and tries to understand the question therefore it can be categorized as Polya's understanding phase. In Scheonfeld's episodes line 1 falls under reading and lines 3-4 falls under analyzing. In Lines 3-4 the student tries to understand the question by summarizing the given information. The basic information he obtains by reading is not sufficient to solve the problem. Although lines 1-4 is listed under understanding, no proper understanding will be acquired until the student investigates the information.

It is difficult to categorize Lines 5-8 under a single category of Polya's Phases. In these lines the student tries to ' emphasize different parts, examine different details, examine the same detail repeatedly but in different ways , combine the details differently, approach them from different sides' Polya (1973, p. 34) . Thus, this falls under planning. Moreover, the line 'those triangles are similar (line 6)' shows that the student tries to identify details and to 'contact with formally acquired knowledge' (Polya, 1973, p. 34). Furthermore, the student constantly argues with himself ('Isn't that what I want?' line 7) to understand the problem as he is unable to plan the method of solving. Therefore, these lines could be categorized under understanding as well as planning.

In items 9-19 it is difficult to separate the planning and implementation as the child does calculations while devising the plan mentally (thinking loud) . The statements 'so if I construct theâˆš2….. then I should be able to draw this line' (line 9) ,' so I just got to remember how to make this construction' (line 12) and 'the best way to do that is to construct A' illustrates that the student is formulating a plan. Whereas the statements ' 1/âˆš2 - let me see here - ummm. That's ½ plus ½ is 1' (line13) and the hand written workings of the student shows the implementation of the plan.

However, in lines 22-38 and 40-48 the solver 'works for better understanding' indicates understanding and 'examines the same detail repeatedly but in different ways' indicates planning (Polya, 1973, p. 34) and acquires a better understanding of the problem ultimately. But this cannot be listed under any of the above two phases as this makes the student engage in a much complicated analysis than in a direct word problem (where the student is able to understand the problem by examining the given data). Furthermore, in these lines the student is 'playing around'(Frobisher, 1994, p. 164) with the ideas. Hence it is difficult to identify into which category these lines would fall into.

According to Schoenfeld Lines 3- 8 , 22- 38 and 40-48, are categorized as the 'analyzing episode' as this is 'an attempt made to fully understand the problem, to select an appropriate perspective and reformulate the problem in those terms and to introduce for consideration whatever principles or mechanisms might be appropriate' (Schoenfeld, 1997, p. 298). In these lines the student tries mechanisms to approach the problem.

Item 49-53, shows that the student explores better ways of constructing the triangle by investigating the information. Frobisher (1994, p. 164) states ' understanding of a problem only emerges slowly as it is explored' and that 'it is difficult for a pupil to develop an understanding of the problem without first attempting to explore whatever appears appropriate'. In these lines the student 'explore' possibilities. But Polya does not define this type of student behavior. So it cannot be categorized under Ploya's Phases. However these lines are listed under Schoenfeld's exploring episode.

However items

Relating Burton's phases and Scheonfeld's Episodes.

One common observation that is visible throughout this protocol analysis was that when using Burton's phases, 'chunks' of lines could not be categorized into one particular phase. Let us examine the analysis in detail.

Lines 1- 8 , can be recognized as the entry phase. The student reads the problem (line1) and tries to understand the question by 'organizing the information'(L Burton, 1984, p. 26) (line 3) . Furthermore, in lines 4, 7 and 8 the student uses the rubric 'I want to' (Leone Burton, 1984) which clearly defines this is the entry phase. On the other hand, Schoenfeld categorizes line 1 as reading and lines 3-8 as analyzing. The student 'organizing information' and 'exploring the problem' which is listed under entry is similar to 'analysis episode' . Thus, entry phase can be interpreted as a collection of reading and analyzing .

Lines 9-19 can be categorized as the 'attack' . In this section it can be seen that the student gets 'stuck' while trying to find resolutions to the problem and overcomes the situation by getting an answer (Ah huh! Ah huh! - in line 13). Moreover, Schoenfeld categorize these lines under 'planning and implementing'.

Lines 20-24 is the 'review phase' according to Burton and 'verification episode' under Schoenfeld's episodes.

In Line 25-39 and lines 40 -48, the student analyses the information and tries to identify what should be done to solve the question. Burton (1984, p. 38) says in 'attack ' , ' several approaches may be t